

ROBUSTNESS PROPERTIES OF DIMENSIONALITY REDUCTION WITH GAUSSIAN RANDOM MATRICES

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ABSTRACT. In this paper we study the robustness properties of dimensionality reduction with Gaussian random matrices having arbitrarily erased rows. We first study the robustness property against erasure for the almost norm preservation property of Gaussian random matrices by obtaining the optimal estimate of the erasure ratio for a small given norm distortion rate. As a consequence, we establish the robustness property of Johnson-Lindenstrauss lemma and the robustness property of restricted isometry property with corruption for Gaussian random matrices. Secondly, we obtain a sharp estimate for the optimal lower and upper bounds of norm distortion rates of Gaussian random matrices under a given erasure ratio. This allows us to establish the strong restricted isometry property with the almost optimal RIP constants, which plays a central role in the study of phaseless compressed sensing.

1. INTRODUCTION AND MOTIVATIONS

In this paper we are interested in investigating various robustness properties of dimensionality reduction with Gaussian random matrices having arbitrarily erased rows. Then we shall use the results to study the robustness properties of the Johnson-Lindenstrauss lemma and restricted isometry property.

Throughout the paper, $A = (a_{j,k})_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathbb{R}^{m \times n}$ will be a Gaussian random matrix such that each entry $a_{j,k}$ is an independent identically distributed (i.i.d.) random variable under the standard Gaussian/normal distribution $\mathcal{N}(0, 1)$ with zero mean and unit standard deviation. For $T \subseteq \{1, \dots, m\}$, we shall adopt the notation $A_T \in \mathbb{R}^{|T| \times n}$ to denote the $|T| \times n$ sub-matrix of A by keeping the rows of A with the row indices from T , where $|T|$ is the cardinality of the set T . Let $x_0 \in \mathbb{R}^n$ be a fixed vector with $\|x_0\| = 1$, where $\|x_0\|$ is the Euclidean norm of the vector x_0 . For $\epsilon > 0$ and $0 \leq \beta < 1$, we define

$$(1.1) \quad \Omega_{\epsilon, \beta} := \Omega_{\epsilon, \beta}(A, x_0) := \left\{ \left| \frac{1}{|T|} \|A_T x_0\|^2 - 1 \right| \leq \epsilon \text{ for all } T \subseteq \{1, \dots, m\} \text{ satisfying } |T^c| \leq \beta m \right\},$$

where $T^c := \{1, \dots, m\} \setminus T$. For every fixed $\epsilon > 0$, it follows from the definition in (1.1) that $\mathbb{P}(\Omega_{\epsilon, \beta})$ is a decreasing function in terms of β , where the probability is taken over the Gaussian random matrix A .

It is well known in the literature by standard tail-bounds for the chi-squared distribution (e.g., see [1, Lemma 4.1]) that

$$(1.2) \quad \begin{aligned} \mathbb{P}\left\{ \frac{1}{m} \|Ax_0\|^2 > 1 + \epsilon \right\} &< e^{-(\epsilon^2/4 - \epsilon^3/6)m}, \\ \mathbb{P}\left\{ \frac{1}{m} \|Ax_0\|^2 < 1 - \epsilon \right\} &< e^{-(\epsilon^2/4 - \epsilon^3/6)m}, \end{aligned} \quad \forall m \in \mathbb{N}, 0 < \epsilon < 1.$$

Consequently, with high probability, a normalized Gaussian random matrix $\frac{1}{\sqrt{m}}A$ has the following almost norm preservation property:

$$(1.3) \quad \mathbb{P}(\Omega_{\epsilon, 0}) = \mathbb{P}\left\{ \left| \frac{1}{m} \|Ax_0\|^2 - 1 \right| \leq \epsilon \right\} \geq 1 - 2e^{-(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}, 0 < \epsilon < 1.$$

The inequality in (1.3) also implies the Johnson-Lindenstrauss lemma (see [1, 9]). For N points $p_1, \dots, p_N \in \mathbb{R}^n$ and for $0 < \epsilon < 1$, the Johnson-Lindenstrauss lemma says that for $m = O(\frac{\ln N}{\epsilon^2})$, there exists a projection matrix $A \in \mathbb{R}^{m \times n}$ such that the following almost norm preservation property holds:

$$(1.4) \quad (1 - \epsilon)\|p_j - p_k\|^2 \leq \|Ap_j - Ap_k\|^2 \leq (1 + \epsilon)\|p_j - p_k\|^2, \quad \forall 1 \leq j, k \leq N.$$

2010 *Mathematics Subject Classification.* 41A99, 60B20, 94A12, 94A20.

Key words and phrases. Gaussian random matrices, sparse approximation, arbitrary erasure, robustness, almost norm preservation, restricted isometry property with corruption, robust Johnson-Lindenstrauss lemma, strong restricted isometry property.

Research of Bin Han was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Canada Grant No. 05865). Research of Zhiqiang Xu was supported by NSFC grant (11171336, 11422113, 11021101, 11331012) and by National Basic Research Program of China (973 Program 2015CB856000).

To establish the above almost norm preservation property in (1.4), the projection matrix A is often taken to be a random matrix so that the almost norm preservation property in (1.3) holds with high probability. The Johnson-Lindenstrauss lemma is a fundamental technique to reduce the dimensionality of the data and has many applications in information theory, machine learning and algorithms (c.f. [6] and references therein).

In the compressed sensing literature, the restricted isometry property (RIP) matrix is also related to (1.3). For $x \in \mathbb{R}^n$ and $s \in \mathbb{N}$, we say that x is s -sparse if x has no more than s nonzero entries. Under a measurement matrix $A \in \mathbb{R}^{m \times n}$, we have $y := (y_1, \dots, y_m)^\top := Ax$ with m measurements y_1, \dots, y_m . To successfully recover the unknown sparse signal x from the measurement vector y , it is important for the matrix A to satisfy the *restricted isometry property* (RIP) with a small positive RIP constant $0 < \epsilon_s < 1$:

$$(1.5) \quad (1 - \epsilon_s)\|v\|^2 \leq \|Av\|^2 \leq (1 + \epsilon_s)\|v\|^2, \quad \text{for all } s\text{-sparse vectors } v \in \mathbb{R}^n.$$

The above restricted isometry property with a small positive RIP constant ϵ_s is often established by considering A to be a random matrix such as a normalized Gaussian random matrix so that the almost norm preservation property in (1.3) holds with high probability for $\epsilon = \epsilon_s$ (see [3, 8]).

When $\beta > 0$, we suppose that at most βm rows of the Gaussian random matrix A are arbitrarily erased. It is of interest in both theory and application to study how large is the erasure ratio β so that a normalized Gaussian random matrix $\frac{1}{\sqrt{m}}A$ with any arbitrarily erased βm rows still has the almost norm preservation property with high probability. Particularly, we are interested in the following two problems:

Problem 1 : Give $0 < \epsilon < 1$ and $0 < \alpha < 1$, what is the maximum β so that

$$\mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \text{for all } m \in \mathbb{N}.$$

Problem 2 : Given $0 < \beta < 1$ and $\alpha > 0$, what is the minimum ϵ so that

$$\mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - 2\exp(-\alpha m), \quad \text{for all } m \in \mathbb{N}.$$

Let us briefly explain our motivation for considering $\mathbb{P}(\Omega_{\epsilon, \beta})$ with $\beta > 0$ in the setting of Johnson-Lindenstrauss lemma and of compressed sensing. In Johnson-Lindenstrauss lemma, note that each projected vector Ap_j has m entries. The projected vectors are often transmitted through network by m parallel channels, that is, each entry of Ap_j is transmitted through an independent channel in a parallel manner. If some channels are out of work, we can only receive the corrupted projected vectors $A_T p_j$ instead of Ap_j for $j = 1, \dots, N$, where $T \subseteq \{1, \dots, m\}$ is an unknown subset with $|T^c| \leq \beta m$ for some given corruption/erasure ratio $0 < \beta < 1$. Consequently, it is important to first establish the almost norm preservation property with high probability in (1.3) for $\Omega_{\epsilon, \beta}$ with $\beta > 0$. The compressed sensing with corruption considers the problem that a certain portion of the obtained measurements y_1, \dots, y_m are missing or corrupted by sparse noises (e.g., see [11, 12, 17] and references therein). In other words, one can only obtain the measurements $A_T x$ for some unknown subset $T \subseteq \{1, \dots, m\}$ such that $|T^c| \leq \beta m$ for some given corruption/erasure ratio $0 < \beta < 1$. Therefore, it is important that the matrices A_T have the restricted isometry property with a small positive RIP constant ϵ_s for all subsets $T \subseteq \{1, \dots, m\}$ with $|T^c| \leq \beta m$. Particularly, in compressive phase retrieval, to recover sparse signals from the magnitude of the linear measurement, one introduces the concept of *strong RIP* which requires the matrices A_T satisfy the RIP property for all subsets $T \subseteq \{1, \dots, m\}$ with $|T^c| \leq \beta m$ (c.f. [13]). For example, in [13], the authors considered the case $\beta = 1/2$. To achieve this robustness property, it is natural to first establish the almost norm preservation property with high probability by replacing $\Omega_{\epsilon, 0}$ and ϵ in (1.3) with $\Omega_{\epsilon, \beta}$ and ϵ_s , respectively for $\beta > 0$.

We first consider Problem 1. To study how large is the erasure ratio β so that a normalized Gaussian random matrix A with arbitrarily erased βm rows still has the almost norm preservation property with high probability, we introduce a quantity $\beta_{\epsilon, \alpha}^{\max}$ to characterize the largest possible such erasure ratio β with a given fixed high probability rate $\alpha > 0$. For $\epsilon > 0$ and $\alpha > 0$, we define

$$(1.6) \quad \beta_{\epsilon, \alpha}^{\max} := \sup\{0 \leq \beta < 1 : \mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \text{ for all } m \in \mathbb{N}\}.$$

If the above set in the right-hand side of (1.6) is empty, then we simply define $\beta_{\epsilon, \alpha}^{\max} := 0$. Due to (1.3) and $\Omega_{\epsilon, \beta} \subseteq \Omega_{\epsilon, 0}$ for all $0 \leq \beta < 1$, it makes sense for us to only consider $0 < \alpha \leq 1$. The multiplicative constant 3 before $e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}$ in (1.6) is not essential and can be replaced by any absolute constant greater than 2. For simplicity of presentation, we stick to the constant 3 in (1.6).

For $\epsilon > 0$ and $0 \leq \beta < 1$, a closely related notion to $\Omega_{\epsilon,\beta}$ in (1.1) is

$$(1.7) \quad \mathring{\Omega}_{\epsilon,\beta} := \mathring{\Omega}_{\epsilon,\beta}(A, x_0) := \left\{ \left| \frac{1}{m} \|A_T x_0\|^2 - 1 \right| \leq \epsilon \text{ for all } T \subseteq \{1, \dots, m\} \text{ satisfying } |T^c| \leq \beta m \right\}.$$

That is, we used the uniform normalization factor $\frac{1}{m}$ for $\mathring{\Omega}_{\epsilon,\beta}$ in (1.7) instead of the factor $\frac{1}{|T|}$ for $\Omega_{\epsilon,\beta}$ in (1.1). Similar to (1.6), for $\epsilon > 0$ and $\alpha > 0$, we define

$$(1.8) \quad \mathring{\beta}_{\epsilon,\alpha}^{\max} := \sup\{0 \leq \beta < 1 : \mathbb{P}(\mathring{\Omega}_{\epsilon,\beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \text{ for all } m \in \mathbb{N}\}.$$

For the case $\epsilon \rightarrow 0^+$ (that is, ϵ is small for the almost norm preservation property in (1.3)), we have the following result.

Theorem 1.1. *Let A be an $m \times n$ random matrix with independent identically distributed entries obeying $\mathcal{N}(0, 1)$. For every $0 < \alpha < 1$,*

$$(1.9) \quad \left(\frac{1 - \sqrt{\alpha}}{32} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}} < \beta_{\epsilon,\alpha}^{\max} < \left(\frac{2 + 2\epsilon_g}{c_g^2 \epsilon_g} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}}, \quad 0 < \epsilon < \min\left(\frac{1 - \sqrt{\alpha}}{4}, \epsilon_g, 4\epsilon_g^2\right)$$

and

$$(1.10) \quad \left(\frac{1 - \sqrt{\alpha}}{32} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}} < \mathring{\beta}_{\epsilon,\alpha}^{\max} < \left(\frac{1}{4c_g^2} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}}, \quad 0 < \epsilon < \min\left(\frac{1 - \sqrt{\alpha}}{4}, c_g^2 \ln 2, \frac{1}{2c_g^2}\right),$$

where c_g and ϵ_g are absolute positive constants given in (3.15) and (2.10), respectively.

Theorem 1.1 shows that $\beta_{\epsilon,\alpha}^{\max} = O(\frac{\epsilon}{\ln \frac{1}{\epsilon}})$ has the optimal order when ϵ is small enough. Hence, Theorem 1.1 presents a solution to Problem 1 up to a multiplicative constant provided that ϵ is small enough. As a direct consequence of Theorem 1.1 (more precisely, Theorem 3.5), by the standard argument in the literature for proving the Johnson-Lindenstrauss lemma using random matrices, we have the following robust version of the Johnson-Lindenstrauss lemma.

Corollary 1.2. *Let $0 < \alpha < 1$ and $0 < \epsilon < \frac{1 - \sqrt{\alpha}}{4}$. Let $N, m, n \in \mathbb{N}$ such that $m > \frac{\ln(3N(N-1)/2)}{\alpha(\epsilon^2/4 - \epsilon^3/6)}$. Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For any given N points $p_1, \dots, p_N \in \mathbb{R}^n$, with probability at least $1 - \frac{3}{2}N(N-1)e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} > 0$,*

$$(1.11) \quad (1 - \epsilon) \|p_j - p_k\|^2 \leq \frac{1}{m} \|A_T p_j - A_T p_k\|^2 \leq (1 + \epsilon) \|p_j - p_k\|^2, \quad \forall 1 \leq j, k \leq N \text{ and } T \in T_{\epsilon,\alpha},$$

where $T_{\epsilon,\alpha}$ is defined to be

$$(1.12) \quad T_{\epsilon,\alpha} := \left\{ T \subseteq \{1, \dots, m\} : |T^c| \leq m \left(\frac{1 - \sqrt{\alpha}}{32} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}} \right\}.$$

Another consequence of Theorem 3.5 is the following result on the robust restricted isometry property.

Corollary 1.3. *Let $0 < \alpha < 1$ and $0 < \epsilon < \frac{1 - \sqrt{\alpha}}{4}$. Let $s, m, n \in \mathbb{N}$ satisfy $s \ln \frac{24en}{\epsilon s} < \alpha(\epsilon^2/16 - \epsilon^3/24)m - \ln 3$. Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. With probability at least $1 - 3\left(\frac{24en}{\epsilon s}\right)^s e^{-\alpha(\epsilon^2/16 - \epsilon^3/24)m} > 0$,*

$$(1.13) \quad (1 - \epsilon) \|v\|^2 \leq \frac{1}{m} \|A_T v\|^2 \leq (1 + \epsilon) \|v\|^2, \quad \forall s\text{-sparse } v \in \mathbb{R}^n \text{ and } T \in T_{\epsilon/2,\alpha},$$

where $T_{\epsilon/2,\alpha}$ is defined in (1.12).

We now turn to Problem 2, which is also related to erasure robust frames (see [15]). For a given $0 < \beta < 1$, we would like to determine the minimum ϵ so that $\frac{1}{|T|} \|A_T x_0\|^2 \in [1 - \epsilon, 1 + \epsilon]$ with high probability for all $T \subseteq \{1, \dots, d\}$ satisfying $|T^c| \leq \beta m$. For this purpose, we consider the most general case instead of the particular subsets $\Omega_{\epsilon,\beta}$ in (1.1). Recall that $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 1$. For $0 \leq \beta < 1$ and $0 \leq \theta \leq \omega \leq \infty$, we define

$$(1.14) \quad \Omega_{[\theta,\omega],\beta} := \Omega_{[\theta,\omega],\beta}(A, x_0) := \left\{ \frac{1}{|T|} \|A_T x_0\|^2 \in [\theta, \omega] \forall T \subseteq \{1, \dots, m\} \text{ satisfying } |T^c| \leq \beta m \right\}.$$

Obviously, $\Omega_{\epsilon,\beta}$ in (1.1) simply becomes $\Omega_{\epsilon,\beta} = \Omega_{[1-\epsilon, 1+\epsilon],\beta}$. For $0 < \beta < 1$ and $\alpha > 0$, we define

$$(1.15) \quad \theta_{\beta}^{\max}(\alpha) := \sup\{0 \leq \theta \leq \infty : \mathbb{P}(\Omega_{[\theta,\infty],\beta}) \geq 1 - \exp(-\alpha m) \text{ for all } m \in \mathbb{N}\},$$

$$(1.16) \quad \omega_{\beta}^{\min}(\alpha) := \inf\{0 \leq \omega \leq \infty : \mathbb{P}(\Omega_{[0,\omega],\beta}) \geq 1 - \exp(-\alpha m) \text{ for all } m \in \mathbb{N}\},$$

and

$$(1.17) \quad \theta_\beta^{\max} := \sup\{\theta_\beta^{\max}(\alpha) : \alpha > 0\} \quad \text{and} \quad \omega_\beta^{\min} := \inf\{\omega_\beta^{\min}(\alpha) : \alpha > 0\}.$$

A simple observation from the above definitions is that $\theta_\beta^{\max}(\alpha) \leq \theta_\beta^{\max} \leq \omega_\beta^{\min} \leq \omega_\beta^{\min}(\alpha)$ and

$$(1.18) \quad \mathbb{P}(\Omega_{[\theta_\beta^{\max}(\alpha), \omega_\beta^{\min}(\alpha)], \beta}) \geq 1 - 2\exp(-\alpha m), \quad \forall m \in \mathbb{N}.$$

If $0 < \theta_\beta^{\max}(\alpha) \leq \omega_\beta^{\min}(\alpha) < 2$, then Problem 2 is solved with $\epsilon = \max(1 - \theta_\beta^{\max}(\alpha), \omega_\beta^{\min}(\alpha) - 1) > 0$. Similar to (1.14), we define

$$(1.19) \quad \mathring{\Omega}_{[\theta, \omega], \beta} := \left\{ \frac{1}{m} \|A_T x_0\|^2 \in [\theta, \omega] \mid \forall T \subseteq \{1, \dots, m\} \text{ satisfying } |T^c| \leq \beta m \right\}$$

and we can define $\mathring{\theta}_\beta^{\max}, \mathring{\omega}_\beta^{\min}$ similar to $\theta_\beta^{\max}, \omega_\beta^{\min}$, respectively by replacing Ω with $\mathring{\Omega}$.

We now briefly explain why we are interested in $\Omega_{[\theta, \omega], \beta}$. An $m \times n$ matrix A is said to have the *strong restricted isometry property* of sparse order $s \in \mathbb{N}$ and level $[\theta, \omega, \beta]$ with $0 < \theta \leq \omega < 2, 0 \leq \beta < 1$ if

$$(1.20) \quad \theta \|v\|^2 \leq \frac{1}{m} \|A_T v\|^2 \leq \omega \|v\|^2, \quad \forall s\text{-sparse } v \in \mathbb{R}^n \text{ and } T \subseteq \{1, \dots, m\} \text{ with } |T^c| \leq \beta m.$$

The strong restricted isometry property plays a critical role in the study of phaseless compressed sensing in [2, 13]. In [13], the authors investigated the case where $\beta = 1/2$ with showing that the Gaussian matrix has the strong RIP of order s and level $[\theta_0, \omega_0, 1/2]$ with high probability provided $m = O(s \log en)$. Here θ_0 and ω_0 are absolute constants. The original motivation for this work is to extend the result in [13] to the arbitrary $\beta \in [0, 1)$. To show that there indeed exists a measurement matrix A having the strong restricted property of sparse order s and level $[\theta, \omega, \beta]$ in (1.20), the matrix A is often constructed by an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$ and one would like to have $\mathbb{P}(\Omega_{[\theta, \omega], \beta}) > 0$ for $0 < \theta \leq \omega < 2$ with the largest possible θ and the smallest possible ω . That is, if we can prove the inequalities $0 < \mathring{\theta}_\beta^{\max} \leq \mathring{\omega}_\beta^{\min} < 2$, for any θ, ω satisfying $0 < \theta < \mathring{\theta}_\beta^{\max} \leq \mathring{\omega}_\beta^{\min} < \omega < 2$, as we shall prove in Corollary 1.6, (1.20) holds with high probability. Thus, the desired inequalities $0 < \mathring{\theta}_\beta^{\max} \leq \mathring{\omega}_\beta^{\min} < 2$ guarantees the strong restricted isometry property for Gaussian random matrices.

We have the following estimates on the quantities $\theta_\beta^{\max}, \omega_\beta^{\min}$ and $\mathring{\theta}_\beta^{\max}, \mathring{\omega}_\beta^{\min}$.

Theorem 1.4. *Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For $0 < \beta < 1$,*

$$(1.21) \quad \frac{\pi}{6}(1 - \beta)^2 \min\left(\frac{3 - 2\beta}{4(1 - \beta)}, 1\right) \leq \theta_\beta^{\max} \leq \min\left(\frac{\pi}{2}\left(\ln \frac{1}{\beta}\right)^2, 1\right),$$

$$(1.22) \quad \max\left(c_g^2 \ln \frac{2}{1 - \beta}, \frac{\pi}{2}\beta^2\right) \leq \omega_\beta^{\min} \leq 2 \ln \frac{e}{1 - \beta},$$

and

$$(1.23) \quad \frac{\pi}{6}(1 - \beta)^3 \min\left(\frac{3 - 2\beta}{4(1 - \beta)}, 1\right) \leq \mathring{\theta}_\beta^{\max} \leq (1 - \beta) \min\left(\frac{\pi}{2}\left(\ln \frac{1}{\beta}\right)^2, 1\right),$$

$$(1.24) \quad (1 - \beta) \max\left(c_g^2 \ln \frac{2}{1 - \beta}, \frac{\pi}{2}\beta^2\right) \leq \mathring{\omega}_\beta^{\min} \leq 2(1 - \beta) \ln \frac{e}{1 - \beta},$$

where the absolute constant c_g is defined in (3.15).

Theorem 1.4 establishes the strong restricted isometry property for Gaussian random matrices for all $\beta \in [0, 1)$ by $(1 - \beta) \ln \frac{e}{1 - \beta} < 1$ for any $\beta \in (0, 1)$ and by (1.3) for $\beta = 0$. As a direct consequence of Theorem 1.4, we have

$$\frac{\pi}{6} \leq \frac{\theta_\beta^{\max}}{(1 - \beta)^2} \leq 2\pi(\ln 2)^2, \quad c_g^2 \leq \frac{\omega_\beta^{\min}}{\ln \frac{1}{1 - \beta}} \leq 2 + \frac{2}{\ln 2}, \quad \forall 1/2 \leq \beta < 1$$

and

$$\frac{\pi}{6} \leq \frac{\mathring{\theta}_\beta^{\max}}{(1 - \beta)^3} \leq 2\pi(\ln 2)^2, \quad c_g^2 \leq \frac{\mathring{\omega}_\beta^{\min}}{(1 - \beta) \ln \frac{1}{1 - \beta}} \leq 2 + \frac{2}{\ln 2}, \quad \forall 1/2 \leq \beta < 1.$$

Thus, up to multiplicative constants, our estimates in Theorem 1.4 for $\theta_\beta^{\max}, \omega_\beta^{\min}$ and $\mathring{\theta}_\beta^{\max}, \mathring{\omega}_\beta^{\min}$ are optimal as $\beta \rightarrow 1^-$.

As an application of Theorem 1.4 and our analysis in Section 4 for proving Theorem 1.4, we have the following robustness properties of Johnson-Lindenstrauss lemma and restricted isometry property with a given erasure ratio $0 < \beta < 1$.

Corollary 1.5. *Let $0 < \beta < 1$ and $0 < \alpha < \frac{\pi}{12}(1 - \beta)^2 h_\beta$ with $h_\beta := \min(\frac{3}{4} - \frac{1}{2}\beta, 1 - \beta)$. Let $m, n, N \in \mathbb{N}$ such that $m \geq \frac{1}{1-\beta}$ and $m > \frac{1}{\alpha} \ln \frac{1}{N(N-1)}$. Let A be an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For any given N points $p_1, \dots, p_N \in \mathbb{R}^n$, with probability at least $1 - N(N-1)e^{-\alpha m} > 0$,*

$$(1.25) \quad \theta \|p_j - p_k\|^2 \leq \frac{1}{m} \|A_T p_j - A_T p_k\|^2 \leq \omega \|p_j - p_k\|^2, \\ \forall 1 \leq j, k \leq N \text{ and } T \subseteq \{1, \dots, m\} \text{ with } |T^c| \leq \beta m,$$

where $\theta, \omega \in (0, \infty)$ are positive real numbers given by

$$(1.26) \quad \theta := \frac{\pi}{6}(1 - \beta)^2 h_\beta + 2\alpha - 2(1 - \beta)\sqrt{\pi \alpha h_\beta / 3}, \quad \omega := \left(\sqrt{2(1 - \beta - \frac{1}{m}) \ln \frac{e}{1 - \beta - \frac{1}{m}}} + \sqrt{2\alpha} \right)^2.$$

Corollary 1.6. *Let $0 < \beta < 1$ and $0 < \alpha < \frac{\pi}{12}(1 - \beta)^2 h_\beta$ with $h_\beta := \min(\frac{3}{4} - \frac{1}{2}\beta, 1 - \beta)$. Let $m, n, s \in \mathbb{N}$ such that $m \geq \frac{1}{1-\beta}$ and $s \ln \frac{24en}{\epsilon s} < \alpha m - \ln 2$. Let A be an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For any $0 < \epsilon < 1$, with probability at least $1 - 2(\frac{24en}{\epsilon s})^s e^{-\alpha m} > 0$,*

$$(1.27) \quad \theta(1 - \epsilon) \|v\|^2 \leq \frac{1}{m} \|A_T v\|^2 \leq \omega(1 + \epsilon) \|v\|^2, \quad \forall s\text{-sparse } v \in \mathbb{R}^n \text{ and } T \subseteq \{1, \dots, m\} \text{ with } |T^c| \leq \beta m,$$

where the positive real numbers θ and ω are given in (1.26).

It is of interest to extend the main results in this paper from Gaussian random matrices to other random matrices such as sub-Gaussian matrices and circulant matrices (c.f. [14]). If A is the Bernoulli matrix, i.e., $\mathbb{P}(a_{j,k} = 1/\sqrt{m}) = \mathbb{P}(a_{j,k} = -1/\sqrt{m}) = 1/2$. Define 2-sparse vectors $v_1 := (1, 1, 0, \dots, 0)^T \in \mathbb{R}^n$ and $v_2 := (1, -1, 0, \dots, 0)^T \in \mathbb{R}^n$. Then either $\inf \{ \frac{1}{m} \|A_T v_1\| : T \subseteq \{1, \dots, m\}, |T^c| \leq m/2 \} = 0$ or $\sup \{ \frac{1}{m} \|A_T v_2\| : T \subseteq \{1, \dots, m\}, |T^c| \leq m/2 \} = 0$. That is, for any $\theta > 0$, either $\mathbb{P}(\Omega_{[\theta, \infty], 1/2}(A, v_1)) = 0$ or $\mathbb{P}(\Omega_{[\theta, \infty], 1/2}(A, v_2)) = 0$ for all $m \in \mathbb{N}$. As a consequence, the strong restricted isometry property for $\beta = 1/2$ cannot hold for Bernoulli random matrices. This shows that the results and study for sub-Gaussian random matrices will be essentially different to Gaussian random matrices. We shall study random matrices other than Gaussian random matrices elsewhere.

The structure of the paper is as follows. In Section 2, we shall provide some auxiliary results for the proofs of the main results in later sections. In Section 3, we shall study the robustness properties of Gaussian random matrices with arbitrarily erased rows for small distortion rates $\epsilon \rightarrow 0^+$. In particular, we shall prove in Section 3 Theorem 1.1 and a few other results related to Theorem 1.1. In certain sense, we studied in Theorem 1.1 the quantities $\beta_{\epsilon, \alpha}^{\max}$ and $\hat{\beta}_{\epsilon, \alpha}^{\max}$ for the case of small erasure ratios $\epsilon \rightarrow 0^+$. In Section 4, we shall study the robustness properties of Gaussian random matrices with arbitrarily erased rows for a given corruption/erasure ratio $0 < \beta < 1$. In particular, we are interested in the behavior of θ_β^{\max} , ω_β^{\min} and $\hat{\theta}_\beta^{\max}$, $\hat{\omega}_\beta^{\min}$ when $\beta \rightarrow 1^-$. We shall prove in Section 4 Theorem 1.4 and other results related to Theorem 1.4. We shall also show that our result leads to the establishment of the strong restricted isometry property for Gaussian random matrices. As applications of the main results in this paper for dimensionality reduction, in Section 5 we shall prove Corollaries 1.2, 1.3 and Corollaries 1.5, 1.6.

2. AUXILIARY RESULTS

In this section we provide some auxiliary results that will be used in later sections. For $y = (y_1, \dots, y_m)^T \in \mathbb{R}^m$, we define $y_{(1)}, \dots, y_{(m)}$ to be the nonincreasing rearrangements of y_1, \dots, y_m in terms of magnitudes such that $|y_{(1)}| \geq \dots \geq |y_{(m)}|$. Let $m \in \mathbb{N}$. For any $0 \leq \gamma \leq 1$ such that γm is an integer, we define

$$(2.1) \quad T_\gamma := \{T \subseteq \{1, \dots, m\} : |T^c| = \gamma m\}.$$

The following simple observation will facilitate our discussion in later sections.

Lemma 2.1. *For $0 \leq \gamma \leq \beta < 1$ such that both γm and βm are integers,*

$$\min_{T \in T_\beta} \left| \frac{1}{|T|} \|A_T x_0\|^2 - 1 \right| \leq \min_{T \in T_\gamma} \left| \frac{1}{|T|} \|A_T x_0\|^2 - 1 \right| \leq \max_{T \in T_\gamma} \left| \frac{1}{|T|} \|A_T x_0\|^2 - 1 \right| \leq \max_{T \in T_\beta} \left| \frac{1}{|T|} \|A_T x_0\|^2 - 1 \right|.$$

Proof. Let $k_\gamma := \gamma m$ and $k_\beta := \beta m$. By $0 \leq \gamma \leq \beta$, we have $k_\gamma \leq k_\beta$ and it follows from $|y_{(1)}| \geq \dots \geq |y_{(m)}|$ that

$$\begin{aligned} \min_{T \in T_\beta} \frac{\|A_T x_0\|^2}{|T|} &= \frac{y_{(k_\beta+1)}^2 + \dots + y_{(m)}^2}{m - k_\beta} \leq \frac{y_{(k_\gamma+1)}^2 + \dots + y_{(m)}^2}{m - k_\gamma} = \min_{T \in T_\gamma} \frac{\|A_T x_0\|^2}{|T|} \\ &\leq \max_{T \in T_\gamma} \frac{\|A_T x_0\|^2}{|T|} = \frac{y_{(1)}^2 + \dots + y_{(m-k_\gamma)}^2}{m - k_\gamma} \leq \frac{y_{(1)}^2 + \dots + y_{(m-k_\beta)}^2}{m - k_\beta} = \max_{T \in T_\beta} \frac{\|A_T x_0\|^2}{|T|}. \end{aligned}$$

Now the claim follows directly from the above inequalities. \square

The following well-known concentration inequalities for the standard Gaussian/normal distribution (e.g., see [10]) will be used later.

Theorem 2.2. *Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a Lipschitz function with Lipschitz constant 1 satisfying $|f(x) - f(y)| \leq \|x - y\|$ for all $x, y \in \mathbb{R}^m$. For i.i.d. standard Gaussian/normal random variables $X_1, \dots, X_m \sim \mathcal{N}(0, 1)$ and for all $\delta \geq 0$,*

$$(2.2) \quad \mathbb{P}\{f(X_1, \dots, X_m) < \delta + \mathbb{E}[f(X_1, \dots, X_m)]\} \geq 1 - e^{-\delta^2/2},$$

$$(2.3) \quad \mathbb{P}\{f(X_1, \dots, X_m) > -\delta + \mathbb{E}[f(X_1, \dots, X_m)]\} \geq 1 - e^{-\delta^2/2}.$$

As an application of the above result, we have the following result (also c.f. [13]).

Lemma 2.3. *Let y_1, \dots, y_m be i.i.d. Gaussian/normal random variables obeying $\mathcal{N}(0, 1)$. Then for all nonempty subsets $S \subseteq \{1, \dots, m\}$ and $\delta > 0$,*

$$(2.4) \quad \mathbb{P}\left\{\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} < \delta + \mathbb{E}\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2}\right\} \geq 1 - e^{-\delta^2|S|/2}$$

and

$$(2.5) \quad \mathbb{P}\left\{\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} > -\delta + \mathbb{E}\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2}\right\} \geq 1 - e^{-\delta^2|S|/2}.$$

Proof. Define $F_S(x) := \sqrt{\sum_{j \in S} y_{(j)}^2}$. Then it is easy to observe that

$$|F_S(x) - F_S(y)|^2 \leq \sum_{j \in S} (|x_{(j)}| - |y_{(j)}|)^2 \leq \sum_{j=1}^m (|x_{(j)}| - |y_{(j)}|)^2 = \|x\|^2 + \|y\|^2 - 2 \sum_{j=1}^m |x_{(j)} y_{(j)}| \leq \|x - y\|^2,$$

where in the last step we used the rearrangement inequality $\sum_{j=1}^m |x_j y_j| \leq \sum_{j=1}^m |x_{(j)} y_{(j)}|$. Therefore, F_S is a Lipschitz function with Lipschitz constant 1. By Theorem 2.2, we have

$$\mathbb{P}\left\{\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} < \delta + \mathbb{E}\sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2}\right\} = \mathbb{P}\left\{\sqrt{\sum_{j \in S} y_{(j)}^2} < \delta \sqrt{|S|} + \mathbb{E}\sqrt{\sum_{j \in S} y_{(j)}^2}\right\} \geq 1 - e^{-\delta^2|S|/2}.$$

The inequalities in (2.5) can be proved similarly. \square

The following result extends [7, Example 10]. We provide a proof here by modifying the proof of [7, Example 10].

Lemma 2.4. *Let $y_1, \dots, y_m \in \mathbb{R}^m$ be i.i.d. Gaussian/normal random variables obeying $\mathcal{N}(0, 1)$ and define $y_{(1)}, \dots, y_{(m)}$ to be the nonincreasing rearrangements of y_1, \dots, y_m in terms of magnitudes such that $|y_{(1)}| \geq \dots \geq |y_{(m)}|$.*

(i) *For $1 \leq j \leq m$ and $1 \leq p \leq 2$,*

$$(2.6) \quad \sqrt{\frac{\pi}{2}} \frac{m+1-j}{m+1} \leq \mathbb{E}|y_{(j)}| \quad \text{and} \quad \mathbb{E}|y_{(j)}|^p \leq C_p \sum_{\ell=j}^m \frac{1}{\ell} \leq C_p \left(\frac{1}{j} + \ln \frac{m}{j}\right),$$

where C_p is a positive constant (e.g., $C_1 \leq \sqrt{\frac{\pi}{2}}$, $C_2 \leq 2$) depending only on p and given by

$$(2.7) \quad C_p := p \sup_{0 < t < \infty} t^{p-1} \int_t^\infty e^{(t^2-s^2)/2} ds \leq p \left(\frac{\pi}{2}\right)^{1-\frac{p}{2}}.$$

(ii) For $1 \leq k \leq m$,

$$(2.8) \quad \mathbb{E} \left(\sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right) \leq \sqrt{2 \ln \frac{em}{k}}.$$

(iii) Let $0 \leq \gamma < 1$ and $m \in \mathbb{N}$ such that $k := \gamma m \in \mathbb{N}$. Then

$$(2.9) \quad \sqrt{\frac{\pi}{6}} \sqrt{(1-\gamma) \frac{1-\gamma+\frac{1}{2m}}{1+\frac{1}{m}}} \leq \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \leq \sqrt{2 - \frac{2\gamma}{1-\gamma} \ln \frac{1+\frac{1}{m}}{\gamma+\frac{1}{m}}}.$$

Proof. Define $u(t) := \sqrt{\frac{2}{\pi}} \int_t^\infty e^{-s^2/2} ds$. As shown in [7, Example 10],

$$\begin{aligned} \mathbb{E}|y_{(j)}|^p &= \int_0^\infty \mathbb{P}\{|y_{(j)}| > t^{1/p}\} dt = \sum_{\ell=0}^{m-j} \binom{m}{\ell} \int_0^\infty (u(t^{1/p}))^{m-\ell} (1-u(t^{1/p}))^\ell dt \\ &= \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{m-j} \binom{m}{\ell} \int_0^\infty (u(t))^{m-\ell} (1-u(t))^\ell p t^{p-1} e^{t^2/2} (-du(t)). \end{aligned}$$

(i) Since $e^{t^2/2} \geq 1$ for all $t \in \mathbb{R}$, for $p = 1$, by a change of variable $x = u(t)$, as proved in [7, Example 10],

$$E|y_{(j)}| \geq \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{m-j} \binom{m}{\ell} \int_0^1 x^{m-\ell} (1-x)^\ell dx = \sqrt{\frac{\pi}{2}} \frac{m+1-j}{m+1}.$$

For $1 \leq p \leq 2$, by a change of variable $x = u(t)$, we deduce that

$$\begin{aligned} \mathbb{E}|y_{(j)}|^p &= \sum_{\ell=0}^{m-j} \int_0^\infty \sqrt{\frac{\pi}{2}} p t^{p-1} e^{t^2/2} u(t) \binom{m}{\ell} (u(t))^{m-\ell-1} (1-u(t))^\ell (-du(t)) \\ &\leq C_p \sum_{\ell=0}^{m-j} \binom{m}{\ell} \int_0^1 x^{m-\ell-1} (1-x)^\ell dx = C_p \sum_{\ell=0}^{m-j} \binom{m}{\ell} \frac{(m-\ell-1)! \ell!}{m!} \\ &= C_p \sum_{\ell=0}^{m-j} \frac{1}{m-\ell} = C_p \left(\frac{1}{j} + \sum_{\ell=j+1}^m \frac{1}{\ell} \right) \leq C_p \left(\frac{1}{j} + \int_j^m \frac{1}{x} dx \right) = C_p \left(\frac{1}{j} + \ln \frac{m}{j} \right). \end{aligned}$$

It is easy to prove that if $1 \leq p \leq 2$, then $C_p < \infty$. Indeed, define

$$f(t) := t^{p-1} \int_t^\infty e^{(t^2-s^2)/2} ds = t^{p-1} \int_0^\infty e^{-ts-s^2/2} ds \leq t^{p-1} \int_0^\infty e^{-s^2/2} ds = \sqrt{\frac{\pi}{2}} t^{p-1}.$$

We also have

$$f(t) = t^{p-1} \int_0^\infty e^{-ts-s^2/2} ds \leq t^{p-1} \int_0^\infty e^{-ts} ds = t^{p-2}.$$

Therefore, $C_p = p \sup_{0 < t < \infty} f(t) \leq p \sup_{0 < t < \infty} \min(\sqrt{\frac{\pi}{2}} t^{p-1}, t^{p-2}) = p \left(\frac{\pi}{2}\right)^{1-\frac{p}{2}} < \infty$.

(ii) By (2.6) with $p = 2$, we have $C_2 \leq 2$ and

$$\mathbb{E} \left(\sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right) \leq \sqrt{\frac{1}{k} \sum_{j=1}^k \mathbb{E} y_{(j)}^2} \leq \sqrt{\frac{2}{k} \sum_{j=1}^k \sum_{\ell=j}^m \frac{1}{\ell}} = \sqrt{\frac{2}{k} \left(k + k \sum_{\ell=k+1}^m \frac{1}{\ell} \right)} \leq \sqrt{2 + 2 \ln \frac{m}{k}} = \sqrt{2 \ln \frac{em}{k}}$$

by $\sum_{\ell=k+1}^m \frac{1}{\ell} \leq \int_k^m \frac{1}{x} dx = \ln \frac{m}{k}$. This proves (2.8).

(iii) By the Cauchy-Schwarz inequality, we have

$$\sum_{j=k+1}^m \frac{m+1-j}{m+1} |y_{(j)}| \leq \sqrt{\sum_{j=k+1}^m \left(\frac{m+1-j}{m+1}\right)^2} \sqrt{\sum_{j=k+1}^m y_{(j)}^2}.$$

Therefore, it follows from the first inequality in (2.6) that

$$\begin{aligned} \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} &\geq \frac{1}{\sqrt{m-k}} \left(\sum_{j=k+1}^m \left(\frac{m+1-j}{m+1}\right)^2 \right)^{-1/2} \sum_{j=k+1}^m \frac{m+1-j}{m+1} \mathbb{E} |y_{(j)}| \\ &\geq \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{m-k}} \left(\sum_{j=k+1}^m \left(\frac{m+1-j}{m+1}\right)^2 \right)^{-1/2} \sum_{j=k+1}^m \left(\frac{m+1-j}{m+1}\right)^2 \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m \left(\frac{m+1-j}{m+1}\right)^2} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{1}{(m-k)(m+1)^2} \sum_{j=1}^{m-k} j^2} \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\frac{(m-k+1)(2m-2k+1)}{6(m+1)^2}} \geq \sqrt{\frac{\pi}{6}} \sqrt{\frac{(m-k)(m-k+1/2)}{m(m+1)}}, \end{aligned}$$

since $\frac{m-k+1}{m+1} \geq \frac{m-k}{m}$ by $k \geq 0$. By $k = \gamma m$, we proved the left-hand side of (2.9).

On the other hand, it follows from the second inequality in (2.6) with $p = 2$ that

$$\begin{aligned} \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} &\leq \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m \mathbb{E} y_{(j)}^2} \leq \sqrt{\frac{2}{m-k} \sum_{j=k+1}^m \sum_{\ell=j}^m \frac{1}{\ell}} = \sqrt{\frac{2}{m-k} \sum_{\ell=k+1}^m \sum_{j=k+1}^{\ell} \frac{1}{\ell}} \\ &= \sqrt{2 - \frac{2k}{m-k} \sum_{\ell=k+1}^m \frac{1}{\ell}} \leq \sqrt{2 - \frac{2k}{m-k} \ln \frac{m+1}{k+1}}, \end{aligned}$$

since $\sum_{\ell=k+1}^m \frac{1}{\ell} \geq \int_{k+1}^{m+1} \frac{1}{x} dx = \ln \frac{m+1}{k+1}$. By $k = \gamma m$, we proved the right-hand side of (2.9). \square

Let $W_0 : [-e^{-1}, \infty) \rightarrow [-1, \infty)$ and $W_{-1} : [-e^{-1}, 0) \rightarrow (-\infty, -1]$ be the principal and secondary real-valued Lambert W functions such that $W_0(x)e^{W_0(x)} = x$ for all $x \geq -e^{-1}$ and $W_{-1}(x)e^{W_{-1}(x)} = x$ for all $-e^{-1} \leq x < 0$ (see [5]). Note that W_0 is an increasing function while W_{-1} is a decreasing function.

We shall also need the following auxiliary result in the proof of Theorem 3.4 of Section 3.

Lemma 2.5. *For any positive real number $c_g > 0$,*

$$(2.10) \quad \epsilon_g := \max_{x \geq 2} \frac{c_g^2 \ln(2x) - 1}{x - 1} = \frac{c_g^2 \ln \frac{2}{\beta_g} - 1}{\frac{1}{\beta_g} - 1} = \begin{cases} c_g^2 \ln 4 - 1 > 0, & \text{if } c_g^2 \ln \frac{4}{\sqrt{e}} > 1, \\ -c_g^2 W_0(-2e^{-1-1/c_g^2}) > 0, & \text{if } c_g^2 \ln \frac{4}{\sqrt{e}} \leq 1, \end{cases}$$

and

$$(2.11) \quad 0 < \beta_g \leq \frac{1}{2} \quad \text{with} \quad \beta_g := \begin{cases} \frac{1}{2}, & \text{if } c_g^2 \ln \frac{4}{\sqrt{e}} > 1, \\ -W_0(-2e^{-1-1/c_g^2}), & \text{if } c_g^2 \ln \frac{4}{\sqrt{e}} \leq 1. \end{cases}$$

Proof. Let $t_g := 2e^{-1-1/c_g^2}$. If $0 < c_g^2 \ln \frac{4}{\sqrt{e}} \leq 1$, then $0 < t_g < e^{-1}$ and therefore, both $W_0(-t_g)$ and β_g are well defined. Since W_0 is an increasing function, it is also easy to prove that

$$(2.12) \quad -W_0(-t_g) \leq \frac{1}{2} \iff t_g \leq \frac{1}{2}e^{-1/2} \iff c_g^2 \ln \frac{4}{\sqrt{e}} \leq 1.$$

To prove (2.10), we define $f(x) := \frac{c_g^2 \ln(2x) - 1}{x - 1}$ for $x > 1$. Then

$$f'(x) = \frac{g(x)}{x(x-1)^2} \quad \text{with} \quad g(x) := x + c_g^2(x - x \ln(2x) - 1).$$

By $g'(x) = 1 - c_g^2 \ln(2x)$, the function g increases on $(0, \frac{1}{et_g})$ and decreases on $(\frac{1}{et_g}, \infty)$. Note that f' has the same sign as g on $(1, \infty)$. If $t_g \geq e^{-1}$, then $c_g^2 \ln 2 \geq 1$ and $g'(x) \leq 0$ for all $x > 1$. Hence, $f'(x) \leq 0$ for all $x > 1$ and f decreases on $(1, \infty)$. Therefore, $\epsilon_g = \max_{x \geq 2} f(x) = f(2) = f(1/\beta_g)$, since $\beta_g = 1/2$ by $c_g^2 \ln \frac{4}{\sqrt{e}} > c_g^2 \ln 2 \geq 1$. Since g' has only one real root on $(0, \infty)$, if $t_g < e^{-1}$, then g has at most two real roots on $(0, \infty)$ given by $x_1 := \frac{1}{-W_{-1}(-t_g)} < 1 < \frac{1}{-W_0(-t_g)} =: x_2$. Thus, f decreases on (x_2, ∞) and increases on $(1, x_2)$, from which we have $\epsilon_g = \max_{x \geq 2} f(x) = f(\max(2, x_2)) = f(1/\beta_g)$, since $\max(2, x_2) = 1/\beta_g$ by (2.12). \square

3. GAUSSIAN RANDOM MATRICES UNDER ARBITRARY ERASURE OF ROWS FOR SMALL ϵ

In this section we study the robustness property of (normalized) Gaussian random matrices under arbitrary erasure of rows for small $0 < \epsilon < 1$. At the end of this section, we shall prove Theorem 1.1.

3.1. A lower bound for $\beta_{\epsilon, \alpha}^{\max}$. To provide a lower bound for $\beta_{\epsilon, \alpha}^{\max}$ in (1.6), we first prove the following result.

Lemma 3.1. *Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1-\sqrt{\alpha}}{\alpha/2})$, if*

$$(3.1) \quad 0 < \beta \leq \frac{1-\sqrt{\alpha}}{1+\epsilon} \epsilon \quad \text{and} \quad 0 < \beta \ln \frac{e}{\beta} \leq \frac{\epsilon}{2} \left(\sqrt{1-\sqrt{\alpha}} - \sqrt{\frac{\alpha\epsilon}{2}} \right)^2,$$

then

$$(3.2) \quad \mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}.$$

Proof. Set $y := Ax_0$. Since each entry in A is an i.i.d. random variable obeying $\mathcal{N}(0, 1)$ and since $\|x_0\| = 1$, a simple calculation leads to $y_j \sim \mathcal{N}(0, 1)$ for every $j = 1, \dots, m$ and all y_1, \dots, y_m are independent.

Let $m \in \mathbb{N}$ be arbitrary but fixed at this moment. Define $\gamma := \lfloor \beta m \rfloor / m$. Then $0 \leq \gamma \leq \beta$ and $\gamma m \in \mathbb{N}$. If $\gamma = 0$, then $\beta m < 1$ and consequently $|T^c| \leq \beta m$ implies that T^c is the empty set. Therefore, if $\gamma = 0$, then $\Omega_{\epsilon, \beta} = \Omega_{\epsilon, 0}$ and the claim in (3.2) is trivially true by (1.3). Thus, in the following discussion, we assume $\gamma > 0$.

By Lemma 2.1, we conclude that

$$\mathbb{P}(\Omega_{\epsilon, \beta}) = \mathbb{P} \left\{ 1 - \epsilon \leq \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon \mid |T^c| \leq \beta m \right\} = \mathbb{P} \left\{ 1 - \epsilon \leq \min_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq \max_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon \right\},$$

where T_γ is defined in (2.1). Recall that $y_{(1)}, \dots, y_{(m)}$ are nonincreasing rearrangements of y_1, \dots, y_m in terms of magnitude such that $|y_{(1)}| \geq |y_{(2)}| \geq \dots \geq |y_{(m)}|$. Let $T \in T_\gamma$ and $k := |T^c| = \gamma m$. Then $|T| = m - k$. Since $\|y_T\|^2 = \|y\|^2 - \|y_{T^c}\|^2$, we observe that

$$(3.3) \quad \|y\|^2 - (y_{(1)}^2 + \dots + y_{(k)}^2) \leq \|y_T\|^2 \leq \|y\|^2 - (y_{(m-k+1)}^2 + \dots + y_{(m)}^2).$$

Using the above inequalities and noting that $k = \gamma m$, we can easily deduce that

$$\begin{aligned} & \mathbb{P} \left\{ 1 - \epsilon \leq \min_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq \max_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon \right\} \\ &= \mathbb{P} \left\{ 1 - \epsilon \leq \frac{\|y\|^2 - (y_{(1)}^2 + \dots + y_{(k)}^2)}{m - k} \leq \frac{\|y\|^2 - (y_{(m-k+1)}^2 + \dots + y_{(m)}^2)}{m - k} \leq 1 + \epsilon \right\} \\ &= \mathbb{P} \left\{ \frac{y_{(1)}^2 + \dots + y_{(k)}^2}{k} \leq \frac{\|y\|^2}{k} - \frac{(1-\gamma)(1-\epsilon)}{\gamma} \text{ and } \frac{y_{(m-k+1)}^2 + \dots + y_{(m)}^2}{k} \geq \frac{\|y\|^2}{k} - \frac{(1-\gamma)(1+\epsilon)}{\gamma} \right\} \\ &\geq \mathbb{P}(E_0 \cap E_1 \cap E_2) = 1 - \mathbb{P}(E_0^c \cup E_1^c \cup E_2^c) \geq 1 - \mathbb{P}(E_0^c) - \mathbb{P}(E_1^c) - \mathbb{P}(E_2^c), \end{aligned}$$

where

$$(3.4) \quad E_0 := \left\{ 1 - \sqrt{\alpha}\epsilon \leq \frac{\|y\|^2}{m} \leq 1 + \sqrt{\alpha}\epsilon \right\},$$

$$(3.5) \quad E_1 := \left\{ \frac{y_{(1)}^2 + \cdots + y_{(k)}^2}{k} \leq 1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon \right\},$$

$$(3.6) \quad E_2 := \left\{ \frac{y_{(m)}^2 + \cdots + y_{(m-k+1)}^2}{k} \geq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon \right\}.$$

Since $E_0 = \{|\frac{1}{m}\|y\|^2 - 1| \leq \sqrt{\alpha}\epsilon\}$, it follows directly from (1.3) that $\mathbb{P}(E_0) \geq 1 - 2e^{-(\alpha\epsilon^2/4 - \alpha^3/2\epsilon^3/6)m}$. Thus, by $0 < \alpha < 1$, we have

$$\mathbb{P}(E_0^c) \leq 2e^{-(\alpha\epsilon^2/4 - \alpha^3/2\epsilon^3/6)m} = 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} e^{-(1-\sqrt{\alpha})\alpha\epsilon^3m/6} \leq 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}.$$

Next we estimate $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$. By (2.8) of Lemma 2.4 and noting that $k = \gamma m$, we have

$$\mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \sqrt{2 \ln \frac{e}{\gamma}}.$$

For $\delta > 0$, it follows from (2.4) of Lemma 2.3 with $S = \{1, \dots, k\}$ that

$$(3.7) \quad \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\} \geq \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \delta + \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right\} \geq 1 - e^{-\delta^2 \gamma m / 2}.$$

Take

$$(3.8) \quad \delta := \sqrt{1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon} - \sqrt{2 \ln \frac{e}{\gamma}}.$$

We claim that

$$(3.9) \quad \delta \geq \sqrt{\frac{\alpha\epsilon^2}{2\gamma}} > 0.$$

Then it follows from (3.7) and the above inequality that

$$\mathbb{P}(E_1^c) = \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} > \sqrt{1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon} \right\} \leq e^{-\delta^2 \gamma m / 2} \leq e^{-\alpha\epsilon^2 m / 4} \leq e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}.$$

On the other hand, by the first inequality in (3.1) and the fact that $0 < \gamma \leq \beta$, we have

$$1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon \leq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\beta}\epsilon \leq 1 + \epsilon - (1 + \epsilon) = 0,$$

which yields

$$\mathbb{P}(E_2) = \mathbb{P} \left\{ \frac{y_{(m)}^2 + \cdots + y_{(m-k+1)}^2}{k} \geq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma}\epsilon \right\} \geq \mathbb{P}\{y_{(m)}^2 + \cdots + y_{(m-k+1)}^2 \geq 0\} = 1.$$

That is, $\mathbb{P}(E_2^c) = 0$. Putting all estimates together, we complete the proof of (3.2).

We now prove (3.9). By our assumption $0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{\alpha/2}$, we observe that

$$(3.10) \quad f_{\epsilon, \alpha} \geq 0 \quad \text{with} \quad f_{\epsilon, \alpha} := \sqrt{\frac{\epsilon}{2}} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha\epsilon}{2}} \right).$$

Now it is straightforward to check that

$$(3.11) \quad 2x^2 + 2\sqrt{\alpha\epsilon^2}x \leq 2f_{\epsilon, \alpha}^2 + 2\sqrt{\alpha\epsilon^2}f_{\epsilon, \alpha} = (1 - \sqrt{\alpha})\epsilon - \frac{\alpha\epsilon^2}{2}, \quad \text{for all } x \in [0, f_{\epsilon, \alpha}].$$

Note that the function $x \ln \frac{e}{x}$ is an increasing function on the interval $(0, 1]$. Since $0 < \gamma \leq \beta < 1$, by the second inequality in (3.1), we have

$$0 < \sqrt{\gamma \ln \frac{e}{\gamma}} \leq \sqrt{\beta \ln \frac{e}{\beta}} \leq f_{\epsilon, \alpha}.$$

Consequently, plugging $x = \sqrt{\gamma \ln \frac{e}{\gamma}}$ into the inequality in (3.11), we deduce that

$$2\gamma \ln \frac{e}{\gamma} + 2\sqrt{\alpha \epsilon^2} \sqrt{\gamma \ln \frac{e}{\gamma}} \leq (1 - \sqrt{\alpha})\epsilon - \frac{\alpha \epsilon^2}{2}.$$

Since $\gamma > 0$, dividing γ on both sides of the above inequality, we obtain

$$2 \ln \frac{e}{\gamma} + 2\sqrt{\frac{\alpha \epsilon^2}{2\gamma}} \sqrt{2 \ln \frac{e}{\gamma}} \leq \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon - \frac{\alpha \epsilon^2}{2\gamma}.$$

Hence, by $1 - \epsilon \geq 0$, we conclude that

$$\left(\sqrt{2 \ln \frac{e}{\gamma}} + \sqrt{\frac{\alpha \epsilon^2}{2\gamma}} \right)^2 = 2 \ln \frac{e}{\gamma} + 2\sqrt{\frac{\alpha \epsilon^2}{2\gamma}} \sqrt{2 \ln \frac{e}{\gamma}} + \frac{\alpha \epsilon^2}{2\gamma} \leq \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon \leq 1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon.$$

That is, we proved

$$\sqrt{2 \ln \frac{e}{\gamma}} + \sqrt{\frac{\alpha \epsilon^2}{2\gamma}} \leq \sqrt{1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon},$$

which is simply the inequality in (3.9). \square

The following result establishes a lower bound for $\beta_{\epsilon, \alpha}^{\max}$ in (1.6).

Theorem 3.2. *Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{4\alpha})$, if*

$$(3.12) \quad 0 < \beta \leq \frac{(1 - \sqrt{\alpha})\epsilon}{16 \ln \frac{4}{(1 - \sqrt{\alpha})\epsilon}},$$

then (3.2) holds. Consequently,

$$(3.13) \quad \beta_{\epsilon, \alpha}^{\max} > \frac{\epsilon}{\ln \frac{1}{\epsilon}} \frac{1 - \sqrt{\alpha}}{32}, \quad \forall 0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{4}.$$

Proof. We first show that (3.12) implies (3.1) in Lemma 3.1. Since $0 < \epsilon \leq 1$ and $0 < \alpha < 1$, we have $0 < (1 - \sqrt{\alpha})\epsilon \leq 1$. The first inequality in (3.1) follows from (3.12), since

$$0 < \beta \leq t_{\epsilon, \alpha} := \frac{(1 - \sqrt{\alpha})\epsilon}{16 \ln \frac{4}{(1 - \sqrt{\alpha})\epsilon}} \leq \frac{(1 - \sqrt{\alpha})\epsilon}{16 \ln 4} < \frac{(1 - \sqrt{\alpha})\epsilon}{2} \leq \frac{(1 - \sqrt{\alpha})\epsilon}{1 + \epsilon}.$$

Let $f_{\epsilon, \alpha}$ be defined in (3.10). By $0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{4\alpha}$, we have $\frac{\alpha \epsilon}{2} \leq \frac{1 - \sqrt{\alpha}}{8}$ and hence, $f_{\epsilon, \alpha} > 0$ and

$$(3.14) \quad f_{\epsilon, \alpha}^2 = \frac{\epsilon}{2} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha \epsilon}{2}} \right)^2 \geq \frac{\epsilon}{2} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{1 - \sqrt{\alpha}}{8}} \right)^2 = \frac{9 - 4\sqrt{2}}{16} (1 - \sqrt{\alpha})\epsilon.$$

A basic calculation shows that

$$\ln(4ez) < (8 - 4\sqrt{2})z, \quad \forall z \geq \ln 4,$$

from which it is straightforward to deduce, by setting $z = \ln(1/x)$, that

$$\frac{x}{4 \ln(1/x)} \ln \frac{4e \ln(1/x)}{x} < \frac{9 - 4\sqrt{2}}{4} x, \quad \forall 0 < x \leq \frac{1}{4}.$$

Plugging $x := \frac{(1 - \sqrt{\alpha})\epsilon}{4}$ into the above inequality, by (3.14), we conclude that

$$t_{\epsilon, \alpha} \ln \frac{e}{t_{\epsilon, \alpha}} = \frac{x}{4 \ln(1/x)} \ln \frac{4e \ln(1/x)}{x} < \frac{9 - 4\sqrt{2}}{4} x = \frac{9 - 4\sqrt{2}}{16} (1 - \sqrt{\alpha})\epsilon \leq f_{\epsilon, \alpha}^2.$$

Since $\beta \ln \frac{e}{\beta}$ is an increasing function on $(0, 1]$, the second inequality in (3.1) follows from (3.12) by noting that $0 < \beta \ln \frac{e}{\beta} \leq t_{\epsilon, \alpha} \ln \frac{e}{t_{\epsilon, \alpha}} < f_{\epsilon, \alpha}^2$.

Since $t_{\epsilon,\alpha} < \frac{(1-\sqrt{\alpha})\epsilon}{1+\epsilon}$ and $t_{\epsilon,\alpha} \ln \frac{\epsilon}{t_{\epsilon,\alpha}} < f_{\epsilon,\alpha}^2$, there must exist $\delta > 0$ such that (3.1) holds for all $0 < \beta \leq t_{\epsilon,\alpha} + \delta$. Note that $\epsilon \leq \frac{1-\sqrt{\alpha}}{4} < \frac{1-\sqrt{\alpha}}{4\alpha}$ by $0 < \alpha < 1$. We have $\ln \frac{1}{\epsilon} \geq \ln \frac{4}{1-\sqrt{\alpha}}$ and therefore,

$$\beta_{\epsilon,\alpha}^{\max} \geq t_{\epsilon,\alpha} + \delta > \frac{(1-\sqrt{\alpha})\epsilon}{16(\ln \frac{1}{\epsilon} + \frac{4}{1-\sqrt{\alpha}})} \geq \frac{(1-\sqrt{\alpha})\epsilon}{32 \ln \frac{1}{\epsilon}}.$$

This proves (3.13). \square

3.2. An upper bound for $\beta_{\epsilon,\alpha}^{\max}$. We now show that the order $\beta_{\epsilon,\alpha}^{\max} = O(\epsilon/\ln \frac{1}{\epsilon})$ for small ϵ given in Theorem 3.2 is optimal. To do so, we recall a well-known inequality on order statistics (e.g., see item (ii) of [7, Example 10] or see [4, Lemma 3.3.1]): there exists an absolute positive constant c_g (depending only on the Gaussian/normal distribution $\mathcal{N}(0, 1)$) such that

$$(3.15) \quad c_g \sqrt{\ln \frac{2m}{j}} \leq \mathbb{E}|y_{(j)}|, \quad \forall 1 \leq j \leq m/2.$$

We first estimate the quantity $\mathbb{P}(\Omega_{\epsilon,\beta})$.

Lemma 3.3. *Let $m \in \mathbb{N}$ and $0 < \beta \leq 1/2$ such that $\beta m \in \mathbb{N}$. Let $\tilde{\epsilon} > 0$ and $0 < \epsilon \leq 1$. If*

$$(3.16) \quad \delta := c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{(1-\epsilon)\beta + \epsilon + \tilde{\epsilon}} > 0,$$

then

$$(3.17) \quad \mathbb{P}(\Omega_{\epsilon,\beta}) \leq e^{-\tilde{\epsilon}^2 m/4} + e^{-\delta^2 m/2}.$$

Proof. Let $y := Ax_0$ and $k := \beta m$. By Lemma 2.1 and (3.3), we have

$$\begin{aligned} \mathbb{P}(\Omega_{\epsilon,\beta}) &= \mathbb{P} \left\{ \max_{T \in T_\beta} \left| \frac{1}{|T|} \|y_T\|^2 - 1 \right| \leq \epsilon \right\} \leq \mathbb{P} \left\{ \min_{T \in T_\beta} \|y_T\|^2 \geq (1-\epsilon)(1-\beta)m \right\} \\ &= \mathbb{P} \left\{ \|y\|^2 - (y_{(1)}^2 + \cdots + y_{(k)}^2) \geq (1-\epsilon)(1-\beta)m \right\} \\ &\leq \mathbb{P} \left\{ \frac{1}{m} \|y\|^2 > 1 + \tilde{\epsilon} \right\} + \mathbb{P} \{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \}, \end{aligned}$$

where $\eta := (1-\epsilon)\beta + \epsilon + \tilde{\epsilon} > 0$ since $0 < \epsilon \leq 1$. It follows directly from (1.2) that

$$(3.18) \quad \mathbb{P} \left\{ \frac{1}{m} \|y\|^2 > 1 + \tilde{\epsilon} \right\} \leq e^{-\tilde{\epsilon}^2 m/4}.$$

On the other hand,

$$\mathbb{P} \left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\} = \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \sqrt{\frac{\eta}{\beta}} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right\}.$$

Since $k = \beta m \leq m/2$, it follows from the inequality in (3.15) and $|y_{(1)}| \geq \cdots \geq |y_{(k)}|$ that

$$\mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \geq \mathbb{E}(|y_{(k)}|) \geq c_g \sqrt{\ln \frac{2m}{k}} = c_g \sqrt{\ln \frac{2}{\beta}}.$$

Therefore,

$$\begin{aligned} \mathbb{P} \left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\} &= \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \sqrt{\frac{\eta}{\beta}} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right\} \\ &\leq \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \sqrt{\frac{\eta}{\beta}} - c_g \sqrt{\ln \frac{2}{\beta}} \right\} \\ &= \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq -\frac{\delta}{\sqrt{\beta}} \right\}. \end{aligned}$$

By our assumption $\delta > 0$, applying (2.5) in Lemma 2.3 with $S = \{1, \dots, k\}$, we get

$$\mathbb{P}\left\{y_{(1)}^2 + \dots + y_{(k)}^2 \leq \eta m\right\} \leq \mathbb{P}\left\{\sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} - \mathbb{E}\sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq -\frac{\delta}{\sqrt{\beta}}\right\} \leq e^{-\delta^2 k/(2\beta)} = e^{-\delta^2 m/2}.$$

Combining the above inequality with (3.18), we conclude that (3.17) holds. \square

To provide an upper bound for $\beta_{\epsilon, \alpha}^{\max}$ in (1.6), here we introduce a related quantity. For $\epsilon > 0$, $\alpha > 0$ and $C > 0$, we define

$$(3.19) \quad \beta_{\epsilon, \alpha, C}^{\max} := \sup\{0 < \beta < 1 : \mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \text{ for sufficiently large } m \in \mathbb{N}\}.$$

If the above set on the right-hand side of (1.6) is empty, then we simply define $\beta_{\epsilon, \alpha, C}^{\max} := 0$. Trivially, $\beta_{\epsilon, \alpha}^{\max} \leq \beta_{\epsilon, \alpha, 3}^{\max}$ for all $\epsilon > 0$ and $\alpha > 0$.

Theorem 3.4. *Let c_g and ϵ_g be the absolute positive constants defined in (3.15) and (2.10). Then*

$$(3.20) \quad \beta_{\epsilon, \alpha, C}^{\max} < \frac{(1 + \epsilon_g^{-1})\epsilon}{c_g^2 \ln \frac{2\epsilon_g}{\epsilon}}, \quad \forall 0 < \epsilon < \min(1, \epsilon_g), \alpha > 0, C > 0.$$

Proof. If $\beta_{\epsilon, \alpha, C}^{\max} = 0$, the claim is trivially true. Hence, we assume $\beta_{\epsilon, \alpha, C}^{\max} > 0$. We first prove that

$$(3.21) \quad f_{\epsilon}(\beta) := c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{(1 - \epsilon)\beta + \epsilon} \leq 0 \quad \forall 0 < \beta < \min(\frac{1}{2}, \beta_{\epsilon, \alpha, C}^{\max}).$$

By the continuity of the function f_{ϵ} , it suffices to prove (3.21) under the extra assumption that β is a rational number. Suppose that (3.21) fails for some rational number β such that $0 < \beta < \min(\frac{1}{2}, \beta_{\epsilon, \alpha, C}^{\max})$. Then there exists $\tilde{\epsilon} > 0$ such that

$$\delta = c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{(1 - \epsilon)\beta + \epsilon + \tilde{\epsilon}} > 0,$$

where δ is also defined in (3.16). Consequently, by Lemma 3.3,

$$(3.22) \quad \mathbb{P}(\Omega_{\epsilon, \beta}) \leq e^{-\tilde{\epsilon}^2 m/4} + e^{-\delta^2 m/2}$$

provided $\beta m \in \mathbb{N}$. On the other hand, by the definition of $\beta_{\epsilon, \alpha, C}^{\max}$ and $0 < \beta < \beta_{\epsilon, \alpha, C}^{\max}$,

$$(3.23) \quad \mathbb{P}(\Omega_{\epsilon, \beta}) \geq 1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \quad \text{for sufficiently large } m \in \mathbb{N}.$$

Consequently, combining (3.22) and (3.23), we have

$$1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \leq \mathbb{P}(\Omega_{\epsilon, \beta}) \leq e^{-\tilde{\epsilon}^2 m/4} + e^{-\delta^2 m/2}$$

for sufficiently large $m \in \mathbb{N}$ satisfying $\beta m \in \mathbb{N}$. Since β is a rational number, there are infinitely many sufficiently large $m \in \mathbb{N}$ satisfying $\beta m \in \mathbb{N}$. Letting such m go to ∞ , we deduce from the above inequality that

$$1 \leq Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} + e^{-\tilde{\epsilon}^2 m/4} + e^{-\delta^2 m/2} \rightarrow 0,$$

which is a contradiction. Therefore, (3.21) must hold.

Define a function

$$F_{\epsilon}(\beta) := c_g^2 \beta \ln \frac{2}{\beta} - ((1 - \epsilon)\beta + \epsilon), \quad \beta > 0.$$

Then it is trivial to see that $f_{\epsilon}(\beta)$ and $F_{\epsilon}(\beta)$ have the same sign on the interval $\beta \in (0, 1)$. As a direct consequence of (3.21), it is straightforward to see that

$$(3.24) \quad \min(\frac{1}{2}, \beta_{\epsilon, \alpha, C}^{\max}) \leq \inf\{0 < \beta < 1/2 : F_{\epsilon}(\beta) > 0\} =: \beta_{\epsilon}.$$

If the above set on the right-hand side is empty, then we simply define $\beta_{\epsilon} = 1/2$. Let β_g and ϵ_g be defined as in Lemma 2.5. Since $0 < \beta_g \leq 1/2$ and $\epsilon_g > 0$, we deduce that

$$F_{\epsilon}(\beta_g) = c_g^2 \beta_g \ln \frac{2}{\beta_g} - ((1 - \epsilon)\beta_g + \epsilon) = (1 - \beta_g)(\epsilon_g - \epsilon) > 0, \quad \forall 0 < \epsilon < \epsilon_g,$$

where we used $c_g^2 \ln \frac{2}{\beta_g} - 1 = \epsilon_g(\frac{1}{\beta_g} - 1)$ by (2.10). Since $\lim_{\beta \rightarrow 0+} F_{\epsilon}(\beta) = -\epsilon < 0$, F_{ϵ} must have a real root inside the interval $(0, \beta_g)$. Hence, $0 < \beta_{\epsilon} < \beta_g \leq 1/2$ and $F_{\epsilon}(\beta_{\epsilon}) = 0$. For $0 < \epsilon < \epsilon_g$,

$$\epsilon = F_{\epsilon}(\beta_{\epsilon}) + \epsilon = (c_g^2 \ln \frac{2}{\beta_{\epsilon}} - 1 + \epsilon)\beta_{\epsilon} \geq (c_g^2 \ln \frac{2}{\beta_g} - 1)\beta_{\epsilon} = \epsilon_g(\frac{1}{\beta_g} - 1)\beta_{\epsilon} \geq \epsilon_g \beta_{\epsilon},$$

by $0 < \beta_g \leq 1/2$. It follows from the above inequality that $\beta_\epsilon \leq \frac{\epsilon}{\epsilon_g}$ for all $0 < \epsilon < \epsilon_g$. Hence, for $0 < \epsilon < \epsilon_g$, it follows from $F_\epsilon(\beta_\epsilon) = 0$ and $0 < \beta_\epsilon \leq \frac{\epsilon}{\epsilon_g}$ that

$$\beta_\epsilon = \frac{\epsilon + (1 - \epsilon)\beta_\epsilon}{c_g^2 \ln \frac{2}{\beta_\epsilon}} \leq \frac{\epsilon + (1 - \epsilon)\frac{\epsilon}{\epsilon_g}}{c_g^2 \ln \frac{2\epsilon_g}{\epsilon}} < \frac{(1 + \epsilon_g^{-1})\epsilon}{c_g^2 \ln \frac{2\epsilon_g}{\epsilon}}.$$

By $0 < \beta_\epsilon < \beta_g \leq 1/2$, we conclude that

$$(3.25) \quad \beta_{\epsilon, \alpha, C}^{\max} = \min(\frac{1}{2}, \beta_{\epsilon, \alpha, C}^{\max}) \leq \beta_\epsilon < \frac{(1 + \epsilon_g^{-1})\epsilon}{c_g^2 \ln \frac{2\epsilon_g}{\epsilon}}, \quad \forall 0 < \epsilon < \min(1, \epsilon_g).$$

This proves (3.20). \square

3.3. An estimate for $\beta_{\epsilon, \alpha}^{\max}$. We now study the relation of the quantity $\beta_{\epsilon, \alpha}^{\max}$ in (1.8) using $\dot{\Omega}_{\epsilon, \beta}$ in (1.7) with a uniform normalization factor $\frac{1}{m}$.

Similar to Lemma 3.3 and Theorem 3.2, we have the following result on the lower bound of $\beta_{\epsilon, \alpha}^{\max}$.

Theorem 3.5. *Let A be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{\alpha/2})$, if $0 < \beta < 1$ satisfies the second inequality in (3.1), i.e.,*

$$(3.26) \quad 0 < \beta \ln \frac{e}{\beta} \leq \frac{\epsilon}{2} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha\epsilon}{2}} \right)^2,$$

then

$$(3.27) \quad \mathbb{P}(\dot{\Omega}_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}.$$

Consequently, under the same conditions as in Theorem 3.2, all the claims in Theorem 3.2 hold with $\beta_{\epsilon, \alpha}^{\max}$ being replaced by $\beta_{\epsilon, \alpha}^{\max}$.

Proof. Let $\gamma := \lfloor \beta m \rfloor / m$ and $k := \gamma m$. Then $\mathbb{P}(\dot{\Omega}_{\epsilon, \beta}) = \mathbb{P}(\dot{\Omega}_{\epsilon, \gamma})$. By $\|y_T\| \leq \|y\|$ and $0 < \alpha < 1$, it follows from (3.3) that

$$\begin{aligned} \mathbb{P}(\dot{\Omega}_{\epsilon, \gamma}) &= \mathbb{P} \left\{ 1 - \epsilon \leq \min_{T \in T_\gamma} \frac{1}{m} \|y_T\|^2 \leq \max_{T \in T_\gamma} \frac{1}{m} \|y_T\|^2 \leq 1 + \epsilon \right\} \\ &\geq \mathbb{P} \left\{ (1 - \epsilon)m \leq \|y\|^2 - (y_{(1)}^2 + \cdots + y_{(k)}^2) \quad \text{and} \quad \frac{1}{m} \|y\|^2 \leq 1 + \sqrt{\alpha}\epsilon \right\} \\ &\geq \mathbb{P} \left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq (1 - \sqrt{\alpha})\epsilon m \quad \text{and} \quad \left| \frac{1}{m} \|y\|^2 - 1 \right| \leq \sqrt{\alpha}\epsilon \right\} \\ &= \mathbb{P}(E_0 \cap E_3) \geq 1 - \mathbb{P}(E_0^c) - \mathbb{P}(E_3^c), \end{aligned}$$

where $E_0 := \{|\frac{1}{m} \|y\|^2 - 1| \leq \sqrt{\alpha}\epsilon\}$ as in (3.4) and $E_3 := \{y_{(1)}^2 + \cdots + y_{(k)}^2 \leq (1 - \sqrt{\alpha})\epsilon m\}$. By (1.3) and $0 < \alpha < 1$, we have

$$\mathbb{P}(E_0^c) \leq 2e^{-(\alpha\epsilon^2/4 - \alpha^{3/2}\epsilon^3/6)m} \leq 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}.$$

Recall from (3.7) that the following inequality holds for any $\delta > 0$:

$$(3.28) \quad \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\} \geq \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \leq \delta + \mathbb{E} \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right\} \geq 1 - e^{-\delta^2 \gamma m / 2}.$$

Set

$$\delta := \sqrt{\frac{(1 - \sqrt{\alpha})\epsilon}{\gamma}} - \sqrt{2 \ln \frac{e}{\gamma}}.$$

Since the function $\beta \ln \frac{e}{\beta}$ is an increasing function on $(0, 1]$, by $0 < \gamma \leq \beta < 1$, we deduce from (3.26) that

$$0 < \gamma \ln \frac{e}{\gamma} \leq \beta \ln \frac{e}{\beta} \leq \frac{\epsilon}{2} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha\epsilon}{2}} \right)^2.$$

Since $\gamma > 0$, dividing $\frac{\gamma}{2}$ on both sides and then taking square root on the above inequality, we see that

$$\sqrt{2 \ln \frac{e}{\gamma}} \leq \frac{\sqrt{\epsilon}}{\gamma} \left(\sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha \epsilon}{2}} \right) = \sqrt{\frac{(1 - \sqrt{\alpha})\epsilon}{\gamma}} - \sqrt{\frac{\alpha \epsilon^2}{2\gamma}},$$

from which it is trivial to see that $\delta \geq \sqrt{\frac{\alpha \epsilon^2}{2\gamma}} > 0$ holds. By the definition of the set E_3 , it follows from (3.28) and $\delta \geq \sqrt{\frac{\alpha \epsilon^2}{2\gamma}} > 0$ that

$$\begin{aligned} \mathbb{P}(E_3^c) &= \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} > \sqrt{\frac{(1 - \sqrt{\alpha})\epsilon}{\gamma}} \right\} = \mathbb{P} \left\{ \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} > \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\} \\ &\leq e^{-\delta^2 \gamma m / 2} \leq e^{-\alpha \epsilon^2 m / 4} \leq e^{-\alpha(\epsilon^2 / 4 - \epsilon^3 / 6)m}. \end{aligned}$$

Therefore, $\mathbb{P}(\mathring{\Omega}_{\epsilon, \beta}) = \mathbb{P}(\mathring{\Omega}_{\epsilon, \gamma}) \geq 1 - \mathbb{P}(E_0^c) - \mathbb{P}(E_3^c) \geq 1 - 3e^{-\alpha(\epsilon^2 / 4 - \epsilon^3 / 6)m}$. This proves (3.27).

It has been proved in the proof of Theorem 3.2 that (3.12), combined with $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{4\alpha})$, implies the conditions in (3.1) with \leq being replaced by $<$. Therefore, all the claims in Theorem 3.2 hold with $\beta_{\epsilon, \alpha}^{\max}$ being replaced by $\mathring{\beta}_{\epsilon, \alpha}^{\max}$. \square

To provide an upper bound for $\mathring{\beta}_{\epsilon, \alpha}^{\max}$, we define $\mathring{\beta}_{\epsilon, \alpha, C}^{\max}$ as in (3.19) with $\Omega_{\epsilon, \beta}$ being replaced by $\mathring{\Omega}_{\epsilon, \beta}$. Trivially, $\mathring{\beta}_{\epsilon, \alpha}^{\max} \leq \mathring{\beta}_{\epsilon, \alpha, 3}^{\max}$ for all $\epsilon > 0$ and $\alpha > 0$.

Theorem 3.6. *Let c_g be the absolute positive constant in (3.15). Then*

$$(3.29) \quad \mathring{\beta}_{\epsilon, \alpha, C}^{\max} \leq \frac{\epsilon}{-c_g^2 W_1(-\epsilon / (2c_g^2))} < \frac{\epsilon}{2c_g^2 \ln \frac{2c_g}{\epsilon}}, \quad \forall 0 < \epsilon < \min(1, c_g^2 \ln 2), \alpha > 0, C > 0.$$

Proof. If $\mathring{\beta}_{\epsilon, \alpha, C}^{\max} = 0$, the claim is trivially true. Hence, we assume $\mathring{\beta}_{\epsilon, \alpha, C}^{\max} > 0$. We first prove that

$$(3.30) \quad g_\epsilon(\beta) := c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{\epsilon} \leq 0 \quad \forall 0 < \beta < \min(\frac{1}{2}, \mathring{\beta}_{\epsilon, \alpha, C}^{\max}).$$

It suffices to prove (3.30) for rational numbers β . Suppose not. Then there exists $\tilde{\epsilon} > 0$ such that $\delta := c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{\epsilon + \tilde{\epsilon}} > 0$. Let $y = Ax_0$ and $k := \beta m$. Then

$$\mathbb{P}(\mathring{\Omega}_{\epsilon, \beta}) \leq \mathbb{P}\{\min_{T \in T_\beta} \|y_T\|^2 \geq (1 - \epsilon)m\} \leq \mathbb{P}\{\frac{1}{m} \|y\|^2 > 1 + \tilde{\epsilon}\} + \mathbb{P}\{y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m\}$$

with $\eta := \epsilon + \tilde{\epsilon}$. The same argument as in Lemma 3.3 yields $\mathbb{P}(\mathring{\Omega}_{\epsilon, \beta}) \leq e^{-\tilde{\epsilon}^2 m / 4} + e^{-\delta^2 m / 2}$. Since $0 < \beta < \mathring{\beta}_{\epsilon, \alpha, C}^{\max}$, the definition of $\mathring{\beta}_{\epsilon, \alpha, C}^{\max}$ implies $\mathbb{P}(\mathring{\Omega}_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2 / 4 - \epsilon^3 / 6)m}$ and the same argument as in Theorem 3.4 leads to a contradiction. Therefore, (3.30) must hold.

Note that g_ϵ is an increasing function on $(0, \frac{2}{e})$ and $\frac{1}{2} < \frac{2}{e}$. By $\frac{\epsilon}{c_g^2} \leq \ln 2$ and the simple fact $-W_{-1}(x) < -2 \ln(-x)$ for all $x \in (-e^{-1}, 0)$, it is easy to conclude from (3.30) that (3.29) must hold. \square

3.4. Proof of Theorem 1.1. We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The left-hand inequality in (1.9) follows directly from (3.13) of Theorem 3.2. Since $0 < \epsilon < 4\epsilon_g^2$, we have $\sqrt{\epsilon} \leq 2\epsilon_g$ and therefore, $\ln \frac{2\epsilon_g}{\epsilon} \geq \ln \frac{\sqrt{\epsilon}}{\epsilon} = \frac{1}{2} \ln \frac{1}{\epsilon}$. By Theorem 3.4, we have

$$\beta_{\epsilon, \alpha}^{\max} \leq \beta_{\epsilon, \alpha, 3}^{\max} < \frac{(1 + \epsilon_g^{-1})\epsilon}{c_g^2 \ln \frac{2\epsilon_g}{\epsilon}} \leq \frac{(1 + \epsilon_g^{-1})\epsilon}{\frac{1}{2} c_g^2 \ln \frac{1}{\epsilon}} = \frac{2 + 2\epsilon_g}{c_g^2 \epsilon_g} \frac{\epsilon}{\ln \frac{1}{\epsilon}}.$$

This proves the right-hand inequality in (1.9).

The left-hand inequality in (1.10) follows directly from Theorem 3.5 and (3.13) of Theorem 3.2. Since $\epsilon \leq \frac{1}{2c_g^2}$, we have $\ln \frac{1}{\epsilon} \geq \ln(2c_g^2)$. Note that $0 < \min(c_g^2 \ln 2, \frac{1}{2c_g^2}) < 1$. Now the right-hand inequality in (1.10) follows directly from Theorem 3.6. \square

4. GAUSSIAN RANDOM MATRICES UNDER ARBITRARY ERASURE OF ROWS FOR GIVEN $0 < \beta < 1$

In this section we study the robustness property of Gaussian random matrices with arbitrarily erased rows for a given corruption/erasure ratio $0 < \beta < 1$ with presenting the proof of Theorem 1.4.

4.1. Estimate $\theta_\beta^{\max}(\alpha)$ and $\omega_\beta^{\min}(\alpha)$. To prove Theorem 1.4, we first estimate $\theta_\beta^{\max}(\alpha)$.

Lemma 4.1. *For $0 < \beta < 1$ and $0 < \alpha < \frac{\pi}{12}(1-\beta)^2 h_\beta$ with $h_\beta := \min(\frac{3}{4} - \frac{1}{2}\beta, 1-\beta)$,*

$$(4.1) \quad 0 < \frac{\pi}{6}(1-\beta)h_\beta + \frac{2\alpha}{1-\beta} - 2\sqrt{\pi\alpha h_\beta/3} \leq \theta_\beta^{\max}(\alpha) \leq \min\left(\frac{\pi}{2}\left(\ln\frac{1}{\beta}\right)^2, 1\right).$$

Proof. Define $\gamma := \lfloor \beta m \rfloor / m$ and $k := \gamma m$. Let $y := (y_1, \dots, y_m)^\top := Ax_0$. Since $\gamma m = \lfloor \beta m \rfloor$, by Lemma 2.1 and (3.3), for $\theta \geq 0$, we have

$$(4.2) \quad \mathbb{P}(\Omega_{[\theta, \infty], \beta}) = \mathbb{P}\left\{\min_{T \in T_\gamma} \frac{1}{|T|} \|y_T\|^2 \geq \theta\right\} = \mathbb{P}\left\{\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta}\right\}.$$

By the left-hand inequality in (2.9) and $0 \leq \gamma \leq \beta < 1$, we have

$$\begin{aligned} \mathbb{E}\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} &\geq \sqrt{\frac{\pi}{6}} \sqrt{(1-\gamma) \frac{1-\gamma + \frac{1}{2m}}{1 + \frac{1}{m}}} \geq \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta) \frac{1-\beta + \frac{1}{2m}}{1 + \frac{1}{m}}} \\ &\geq \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)} \sqrt{\inf_{0 < x \leq 1} \frac{1-\beta + \frac{1}{2}x}{1+x}} = \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta}. \end{aligned}$$

Therefore, for $0 < \beta < 1$, by (4.2) and (2.5) of Lemma 2.3 with $\delta := \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta} - \sqrt{\theta}$, if $\delta \geq \sqrt{\frac{2\alpha}{1-\beta}} > 0$, then we have

$$\begin{aligned} \mathbb{P}(\Omega_{[\theta, \infty], \beta}) &= \mathbb{P}\left\{\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} - \mathbb{E}\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta} - \mathbb{E}\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2}\right\} \\ &\geq \mathbb{P}\left\{\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} - \mathbb{E}\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta} - \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta}\right\} \\ &\geq 1 - e^{-\delta^2(m-k)/2} \geq 1 - e^{-\alpha \frac{m-k}{1-\beta}} = 1 - e^{-\frac{1-\gamma}{1-\beta} \alpha m} \geq 1 - e^{-\alpha m}, \end{aligned}$$

since $\frac{1-\gamma}{1-\beta} \geq 1$ by $0 \leq \gamma \leq \beta < 1$. This shows that if $\sqrt{\theta} \leq \sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta} - \sqrt{\frac{2\alpha}{1-\beta}}$, then

$$(4.3) \quad \mathbb{P}(\Omega_{[\theta, \infty], \beta}) \geq 1 - e^{-\alpha m}$$

for all $m \in \mathbb{N}$. Since $0 < \alpha < \frac{\pi}{12}(1-\beta)^2 h_\beta$, we have $\sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta} - \sqrt{\frac{2\alpha}{1-\beta}} > 0$. Consequently, by the definition of $\theta_\beta^{\max}(\alpha)$, we conclude that

$$\theta_\beta^{\max}(\alpha) \geq \left(\sqrt{\frac{\pi}{6}} \sqrt{(1-\beta)h_\beta} - \sqrt{\frac{2\alpha}{1-\beta}}\right)^2 = \frac{\pi}{6}(1-\beta)h_\beta + \frac{2\alpha}{1-\beta} - 2\sqrt{\pi\alpha h_\beta/3} > 0.$$

This proves the left-hand side of (4.1).

We now estimate the upper bound for $\theta_\beta^{\max}(\alpha)$. By the second inequality in (2.6) with $p = 1$, we have

$$(4.4) \quad \mathbb{E}\sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \leq \mathbb{E}|y_{(k+1)}| \leq \sqrt{\frac{\pi}{2}} \left(\frac{1}{k+1} + \ln \frac{m}{k+1}\right) \leq \sqrt{\frac{\pi}{2}} \ln \frac{m}{k} = \sqrt{\frac{\pi}{2}} \ln \frac{1}{\gamma},$$

where we used the basic inequality $\frac{1}{k+1} \leq \ln(1 + \frac{1}{k})$ for all $k > 0$. Suppose that (4.3) holds for sufficiently large $m \in \mathbb{N}$. For convenience, we only consider the case that β is rational, since the general result follows from the fact that the rational numbers are dense in \mathbb{R} . We assume that $m \in \mathbb{N}$ is sufficiently large and

satisfies $m\beta \in \mathbb{N}$, that is, we have $\gamma = \beta$. Note that $k = \gamma m = \beta m$. By (4.2) and (4.4), applying (2.4) of Lemma 2.3 with $\delta := \sqrt{\theta} - \sqrt{\frac{\pi}{2}} \ln \frac{1}{\beta} > 0$, we have

$$\begin{aligned} \mathbb{P}(\Omega_{[\theta, \infty], \beta}) &= \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \right\} \\ &\leq \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta} - \sqrt{\frac{\pi}{2}} \ln \frac{1}{\beta} \right\} \\ &\leq e^{-\delta^2(m-k)/2} = e^{-\delta^2(1-\beta)m/2}. \end{aligned}$$

Consequently, if $\sqrt{\theta} > \sqrt{\frac{\pi}{2}} \ln \frac{1}{\beta}$ and if (4.3) holds for sufficiently large $m \in \mathbb{N}$, then the above inequalities imply

$$1 - e^{-\alpha m} \leq \mathbb{P}(\Omega_{[\theta, \infty], \beta}) \leq e^{-\delta^2(1-\beta)m/2},$$

which cannot be true for sufficiently large m since $\alpha > 0$ and $\delta > 0$. This proves that $\theta_\beta^{\max}(\alpha) \leq \frac{\pi}{2}(\ln \frac{1}{\beta})^2$. Also, it is trivial to see that

$$\mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \leq \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m \mathbb{E} y_{(j)}^2} \leq \sqrt{\frac{1}{m} \sum_{j=1}^m \mathbb{E} y_{(j)}^2} = 1.$$

The above same argument shows that $\theta_\beta^{\max}(\alpha) \leq 1$. This proves the upper bound of (4.1). \square

We next estimate $\omega_\beta^{\min}(\alpha)$.

Lemma 4.2. *For $0 < \beta < 1$ and $\alpha > 0$,*

$$(4.5) \quad \max \left(c_g^2 \ln \frac{2}{1-\beta}, \frac{\pi}{2} \beta^2 \right) \leq \omega_\beta^{\min}(\alpha) \leq 2 \ln \frac{e}{1-\beta} + \frac{2\alpha}{1-\beta} + 4 \sqrt{\frac{\alpha}{1-\beta} \ln \frac{e}{1-\beta}}.$$

Proof. Define $\gamma := \lfloor \beta m \rfloor / m$ and $k := \gamma m$. Let $y := (y_1, \dots, y_m)^\top := Ax_0$. Since $\gamma m = \lfloor \beta m \rfloor$, by Lemma 2.1 and (3.3), for $\omega \geq 0$, we have

$$(4.6) \quad \mathbb{P}(\Omega_{[0, \omega], \beta}) = \mathbb{P} \left\{ \max_{T \in T_\gamma} \frac{1}{|T|} \|y_T\|^2 \leq \omega \right\} = \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega} \right\}.$$

By (2.8), we have

$$\mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{2 \ln \frac{em}{m-k}} = \sqrt{2 \ln \frac{e}{1-\gamma}}.$$

By (4.6) and the above estimate, applying (2.4) of Lemma 2.3 with $\delta := \sqrt{\omega} - \sqrt{2 \ln \frac{e}{1-\gamma}} \geq \sqrt{\frac{2\alpha}{1-\gamma}} > 0$, we have

$$\begin{aligned} \mathbb{P}(\Omega_{[0, \omega], \beta}) &= \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \right\} \\ (4.7) \quad &\geq \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega} - \sqrt{2 \ln \frac{e}{1-\gamma}} \right\} \\ &\geq 1 - e^{-\delta^2(m-k)/2} \geq 1 - e^{-\alpha \frac{m-k}{1-\gamma}} = 1 - e^{-\alpha m}. \end{aligned}$$

If $\sqrt{\omega} \geq \sqrt{\frac{2\alpha}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}}$, then we have $\sqrt{\omega} \geq \sqrt{\frac{2\alpha}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}} \geq \sqrt{\frac{2\alpha}{1-\gamma}} + \sqrt{2 \ln \frac{e}{1-\gamma}}$ by $0 \leq \gamma \leq \beta < 1$ and the above inequality shows that

$$(4.8) \quad \mathbb{P}(\Omega_{[0, \omega], \beta}) \geq 1 - e^{-\alpha m}$$

holds for all $m \in \mathbb{N}$. Therefore, we proved

$$\omega_{\beta}^{\min}(\alpha) \leq \left(\sqrt{\frac{2\alpha}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}} \right)^2 = 2 \ln \frac{e}{1-\beta} + \frac{2\alpha}{1-\beta} + 4 \sqrt{\frac{\alpha}{1-\beta} \ln \frac{e}{1-\beta}}.$$

This proves the right-hand side inequality in (4.5).

Without loss of generality, we assume that β is a rational number and m is sufficiently large satisfying $\beta m \in \mathbb{N}$. Thus, $\gamma = \beta$ and $k = \beta m$. We consider two cases of β . Suppose that $1/2 \leq \beta < 1$. Then $m - k = (1 - \beta)m \leq m/2$ and by (3.15), we have

$$\mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \geq \mathbb{E} |y_{(m-k)}| \geq c_g \sqrt{\ln \frac{2m}{m-k}} = c_g \sqrt{\ln \frac{2}{1-\beta}}.$$

By (4.6) and the above inequality, applying (2.5) of Lemma 2.3 with $\delta := c_g \sqrt{\ln \frac{2}{1-\beta}} - \sqrt{\omega} > 0$ and $S = \{1, \dots, m-k\}$, we have

$$\begin{aligned} \mathbb{P}(\Omega_{[0, \omega], \beta}) &= \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \right\} \\ &\leq \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega} - c_g \sqrt{\ln \frac{2}{1-\beta}} \right\} \\ &= \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} - \mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq -\delta \right\} \\ &\leq e^{-\delta^2(m-k)/2} = e^{-\delta^2(1-\beta)m/2}. \end{aligned}$$

Consequently, if (4.8) holds for sufficiently large $m \in \mathbb{N}$, then

$$1 - e^{-\alpha m} \leq \mathbb{P}(\Omega_{[0, \omega], \beta}) \leq e^{-\delta^2(1-\beta)m/2},$$

which cannot be true when m is sufficiently large. This proves that $\omega_{\beta}^{\min}(\alpha) \geq c_g^2 \ln \frac{2}{1-\beta}$ provided that $1/2 \leq \beta < 1$.

Suppose that $0 < \beta < 1/2$. By (2.6), we have

$$\mathbb{E} \sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \geq \mathbb{E} |y_{(m-k)}| \geq \sqrt{\frac{\pi}{2}} \frac{k+1}{m+1} \geq \sqrt{\frac{\pi}{2}} \frac{k}{m} = \sqrt{\frac{\pi}{2}} \beta.$$

The above same argument shows that $\omega_{\beta}^{\min}(\alpha) \geq \frac{\pi}{2} \beta^2$ provided that $0 < \beta < 1/2$. This proves the left-hand side inequality in (4.5). \square

4.2. Estimate $\hat{\theta}_{\beta}^{\max}(\alpha)$ and $\hat{\omega}_{\beta}^{\min}(\alpha)$. As a direct consequence of Lemma 4.1, we have

Corollary 4.3. For $0 < \beta < 1$ and $0 < \alpha < \frac{\pi}{12}(1-\beta)^2 h_{\beta}$ with $h_{\beta} := \min(\frac{3}{4} - \frac{1}{2}\beta, 1-\beta)$,

$$(4.9) \quad 0 < \frac{\pi}{6}(1-\beta)^2 h_{\beta} + 2\alpha - 2(1-\beta) \sqrt{\pi \alpha h_{\beta}/3} \leq \hat{\theta}_{\beta}^{\max}(\alpha) \leq (1-\beta) \min \left(\frac{\pi}{2} \left(\ln \frac{1}{\beta} \right)^2, 1 \right).$$

Proof. Define $\gamma := \lfloor \beta m \rfloor / m$ and $k := \gamma m$. Let $y := (y_1, \dots, y_m)^{\top} := Ax_0$. Then $0 \leq \gamma \leq \beta < 1$. By the definition of $\hat{\Omega}_{[\theta, \infty], \beta}$, we have

$$\mathbb{P}(\hat{\Omega}_{[\theta, \infty], \beta}) = \mathbb{P} \left\{ \sqrt{\frac{1}{m} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\theta} \right\} = \mathbb{P} \left\{ \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \geq \sqrt{\frac{\theta}{1-\gamma}} \right\} = \mathbb{P}(\Omega_{[\frac{\theta}{1-\gamma}, \infty], \beta}) \geq \mathbb{P}(\Omega_{[\frac{\theta}{1-\beta}, \infty], \beta}),$$

where we used $\frac{1}{1-\gamma} \leq \frac{1}{1-\beta}$ by $0 \leq \gamma \leq \beta < 1$. Consequently, for all $0 \leq \theta < (1-\beta)\theta_\beta^{\max}(\alpha)$, we have $\frac{\theta}{1-\beta} < \theta_\beta^{\max}(\alpha)$ and by the definition of $\theta_\beta^{\max}(\alpha)$, we have

$$1 - e^{-\alpha m} \leq \mathbb{P}(\Omega_{[\frac{\theta}{1-\beta}, \infty], \beta}) \leq \mathbb{P}(\mathring{\Omega}_{[\theta, \infty], \beta}), \quad \forall m \in \mathbb{N}.$$

For $0 < \beta < 1$ and $0 < \alpha \leq \frac{\pi}{12}(1-\beta)^2 h_\beta$, it follows from the above inequality and Lemma 4.1 that

$$\mathring{\theta}_\beta^{\max}(\alpha) \geq (1-\beta)\theta_\beta^{\max}(\alpha) \geq \frac{\pi}{6}(1-\beta)^2 h_\beta + 2\alpha - 2(1-\beta)\sqrt{\pi\alpha h_\beta/3} > 0.$$

This proves the left-hand side inequality in (4.9).

Note that we proved the upper bound of $\theta_\beta^{\max}(\alpha)$ in (4.1) of Lemma 4.1 by assuming that β is rational and m is sufficiently large satisfying $\beta m \in \mathbb{N}$. For such β and m , we have $\gamma = \beta$ and consequently, the same proof of Lemma 4.1 yields

$$\mathring{\theta}_\beta^{\max}(\alpha) = (1-\beta)\theta_\beta^{\max}(\alpha) \leq (1-\beta) \min\left(\frac{\pi}{2}(\ln \frac{1}{\beta})^2, 1\right).$$

This proves the right-hand side inequality in (4.9). \square

With the help of Lemma 4.2, we have the following result.

Corollary 4.4. *For $0 < \beta < 1$ and $\alpha > 0$,*

$$(4.10) \quad (1-\beta) \max\left(c_g^2 \ln \frac{2}{1-\beta}, \frac{\pi}{2}\beta^2\right) \leq \mathring{\omega}_\beta^{\min}(\alpha)$$

and

$$(4.11) \quad \mathbb{P}(\mathring{\Omega}_{[0, \omega], \beta}) \geq 1 - e^{-\alpha m}, \quad \forall \omega \geq \left(\sqrt{2(1-\gamma) \ln \frac{e}{1-\gamma}} + \sqrt{2\alpha}\right)^2, m \in \mathbb{N} \quad \text{with } \gamma := \lfloor \beta m \rfloor / m.$$

Proof. Define $\gamma := \lfloor \beta m \rfloor / m$ and $k := \gamma m$. Let $y := (y_1, \dots, y_m)^\top := Ax_0$. By the definition of $\mathring{\Omega}_{[0, \omega], \beta}$,

$$\mathbb{P}(\mathring{\Omega}_{[0, \omega], \beta}) = \mathbb{P}\left\{\sqrt{\frac{1}{m} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\omega}\right\} = \mathbb{P}\left\{\sqrt{\frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2} \leq \sqrt{\frac{\omega}{1-\gamma}}\right\} = \mathbb{P}(\Omega_{[0, \frac{\omega}{1-\gamma}], \beta}).$$

Note that we proved the lower bound of $\omega_\beta^{\min}(\alpha)$ in (4.5) of Lemma 4.2 by assuming that β is rational and m is sufficiently large satisfying $\beta m \in \mathbb{N}$. For such β and m , we have $\gamma = \beta$ and the left-hand side inequality in (4.10) follows directly from the same proof of Lemma 4.2.

As proved in (4.7), if $\delta := \sqrt{\frac{\omega}{1-\gamma}} - \sqrt{2 \ln \frac{e}{1-\gamma}} \geq \sqrt{\frac{2\alpha}{1-\gamma}} > 0$, then

$$(4.12) \quad \mathbb{P}(\mathring{\Omega}_{[0, \omega], \beta}) = \mathbb{P}(\Omega_{[0, \frac{\omega}{1-\gamma}], \beta}) \geq 1 - e^{-\delta^2(m-k)/2} \geq 1 - e^{-\alpha m}, \quad \forall m \in \mathbb{N} \quad \text{with } \gamma := \lfloor \beta m \rfloor / m.$$

This proves the inequality in (4.11). \square

To estimate the upper bound of $\mathring{\omega}_\beta^{\min}$, we need the following lemma.

Lemma 4.5. *Let $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 1$ and $m \in \mathbb{N}$. Let A be an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For $0 < \beta < 1$ and $\alpha > 0$, let $\omega_{\beta, m}(\alpha)$ be the smallest $\omega \geq 0$ such that*

$$\mathbb{P}(\mathring{\Omega}_{[0, \omega], \beta}) = \mathbb{P}\left\{\sup_{|T^c| \leq \beta m} \frac{1}{m} \|A_T x_0\|^2 \leq \omega\right\} \geq 1 - e^{-\alpha m}.$$

Then $\omega_{\beta, m}(\alpha) > 0$ for all $\alpha > 0$ and $\omega_{\beta, m} := \lim_{\alpha \rightarrow 0^+} \omega_{\beta, m}(\alpha) = 0$.

Proof. Suppose that $\omega_{\beta, m}(\alpha) = 0$. Then $\mathbb{P}(\mathring{\Omega}_{[0, 0], \beta}) \geq 1 - e^{-\alpha m}$, which is a contradiction to $\mathbb{P}(\mathring{\Omega}_{[0, 0], \beta}) = 0$ and $\alpha > 0$. Therefore, we must have $\omega_{\beta, m}(\alpha) > 0$. Note that $\omega_{\beta, m}(\alpha)$ is an increasing function of α . By $\mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}(\alpha)], \beta}) \geq 1 - e^{-\alpha m}$, we have $\mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}(\alpha)], \beta}^c) \leq e^{-\alpha m}$. Then

$$\mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}], \beta}^c) = \lim_{\alpha \rightarrow 0^+} \mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}(\alpha)], \beta}^c) \leq \lim_{\alpha \rightarrow 0^+} e^{-\alpha m} = 1.$$

Define $y = (y_1, \dots, y_m)^\top := Ax_0$. By $0 < \beta < 1$, we see that $T \subseteq \{1, \dots, m\}$ with $|T^c| \leq \beta m$ implies $|T| \geq 1$. If $\omega_{\beta, m} > 0$, it is trivial to see that

$$\mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}], \beta}^c) \leq \mathbb{P}\{|y_1|^2 + \dots + |y_m|^2 > m\omega_{\beta, m}\} < 1,$$

which is a contradiction to $\mathbb{P}(\mathring{\Omega}_{[0, \omega_{\beta, m}], \beta}^c) = 1$. This proves $\omega_{\beta, m} = \lim_{\alpha \rightarrow 0^+} \omega_{\beta, m}(\alpha) = 0$. \square

4.3. Proof of Theorem 1.4. We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Observe that $\theta_\beta^{\max} = \lim_{\alpha \rightarrow 0^+} \theta_\beta^{\max}(\alpha)$ and $\omega_\beta^{\min} = \lim_{\alpha \rightarrow 0^+} \omega_\beta^{\min}(\alpha)$. Taking $\alpha \rightarrow 0^+$ in (4.1) of Lemma 4.1, we have

$$(4.13) \quad \frac{\pi}{6}(1-\beta)^2 \min\left(\frac{3-2\beta}{4(1-\beta)}, 1\right) \leq \theta_\beta^{\max} \leq \min\left(\frac{\pi}{2}\left(\ln \frac{1}{\beta}\right)^2, 1\right).$$

This proves (1.21). Taking $\alpha \rightarrow 0^+$ in (4.5) of Lemma 4.2, we have

$$(4.14) \quad \max\left(c_g^2 \ln \frac{2}{1-\beta}, \frac{\pi}{2}\beta^2\right) \leq \omega_\beta^{\min} \leq 2 \ln \frac{e}{1-\beta}.$$

This proves (1.22). Similarly, taking $\alpha \rightarrow 0^+$ in (4.9) of Corollary 4.3, we have

$$(4.15) \quad \frac{\pi}{6}(1-\beta)^3 \min\left(\frac{3-2\beta}{4(1-\beta)}, 1\right) \leq \dot{\theta}_\beta^{\max} \leq (1-\beta) \min\left(\frac{\pi}{2}\left(\ln \frac{1}{\beta}\right)^2, 1\right).$$

This proves (1.23). We now prove (1.24). Taking $\alpha \rightarrow 0^+$ for the left-hand side of (4.10) in Corollary 4.4, we proved the left-hand side of (1.24). We now prove the right-hand side of (1.24). Take $N \in \mathbb{N}$ such that $N > \frac{1}{\beta}$. Let $\gamma := \lfloor \beta m \rfloor / m$. Then $\beta - \frac{1}{N} \leq \beta - \frac{1}{m} \leq \gamma \leq \beta$ for all $m \geq N$. Since $(1-x) \ln \frac{e}{1-x}$ is a decreasing function on $(0, 1)$, we have $(1-\beta + \frac{1}{N}) \ln \frac{e}{1-\beta + \frac{1}{N}} \geq (1-\gamma) \ln \frac{e}{1-\gamma}$ for all $m \geq N$. Using the same notation as in Lemma 4.5, it follows from (4.11) in Corollary 4.4 that

$$(4.16) \quad \begin{aligned} \omega_\beta(\alpha) &:= \sup_{m \in \mathbb{N}} \omega_{\beta, m}(\alpha) \\ &\leq \max\left(\omega_{\beta, 1}(\alpha), \dots, \omega_{\beta, N-1}(\alpha), \left(\sqrt{2\left(1-\beta + \frac{1}{N}\right) \ln \frac{e}{1-\beta + \frac{1}{N}}} + \sqrt{2\alpha}\right)^2\right). \end{aligned}$$

Taking $\alpha \rightarrow 0^+$ in the above inequality, we deduce from Lemma 4.5 that

$$\begin{aligned} \omega_\beta &:= \lim_{\alpha \rightarrow 0^+} \omega_\beta(\alpha) \leq \lim_{\alpha \rightarrow 0^+} \max\left(\omega_{\beta, 1}(\alpha), \dots, \omega_{\beta, N-1}(\alpha), \left(\sqrt{2\left(1-\beta + \frac{1}{N}\right) \ln \frac{e}{1-\beta + \frac{1}{N}}} + \sqrt{2\alpha}\right)^2\right) \\ &= 2\left(1-\beta + \frac{1}{N}\right) \ln \frac{e}{1-\beta + \frac{1}{N}}. \end{aligned}$$

Since $N > \frac{1}{\beta}$ can be arbitrarily large, combining with (4.10), we proved

$$(1-\beta) \max\left(c_g^2 \ln \frac{2}{1-\beta}, \frac{\pi}{2}\beta^2\right) \leq \dot{\omega}_\beta^{\min} \leq \omega_\beta \leq 2(1-\beta) \ln \frac{e}{1-\beta}.$$

This proves the left-hand side of (1.24). \square

5. PROOFS OF COROLLARIES

We now provide proofs to Corollaries 1.2 and 1.3 as well as the proofs of Corollaries 1.5 and 1.6.

Proof of Corollary 1.2. We assume that all the points p_1, \dots, p_N are distinct. For every $T \in T_{\epsilon, \alpha}$, we have $|T^c| \leq \beta m$ for all $0 \leq \beta \leq \frac{1-\sqrt{\alpha}}{32} \frac{\epsilon}{\ln \frac{1}{\epsilon}}$. By Theorem 3.5, for $j \neq k$, we have

$$\mathbb{P}\left\{\left|\frac{\|A_T p_j - A_T p_k\|^2}{m\|p_j - p_k\|^2} - 1\right| \leq \epsilon \text{ for all } T \in T_{\epsilon, \alpha}\right\} \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}.$$

Here we used the inequality

$$0 < \beta \leq \frac{1 - \sqrt{\alpha}}{32} \frac{\epsilon}{\ln \frac{1}{\epsilon}} \leq \frac{(1 - \sqrt{\alpha})\epsilon}{16 \ln \frac{4}{(1 - \sqrt{\alpha})\epsilon}},$$

provided that $0 < \epsilon < \frac{1 - \sqrt{\alpha}}{4}$. Since there are $\binom{N}{2} = \frac{N(N-1)}{2}$ pairs $\{p_j, p_k\}$ with $j \neq k, j, k = 1, \dots, N$, using union bounds, we conclude that

$$\mathbb{P} \left\{ \left| \frac{\|A_T p_j - A_T p_k\|^2}{m \|p_j - p_k\|^2} - 1 \right| \leq \epsilon \forall T \in T_{\epsilon, \beta}, j \neq k, j, k = 1, \dots, N \right\} \geq 1 - \frac{3N(N-1)}{2} e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} > 0,$$

where we used the assumption of $m > \frac{\ln(3N(N-1)/2)}{\alpha(\epsilon^2/4 - \epsilon^3/6)}$ in the last inequality. This proves that (1.11) holds with probability at least $1 - \frac{3N(N-1)}{2} e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} > 0$. \square

Proof of Corollary 1.3. We slightly modify the argument in [3, Lemma 5.1]. Let $\Lambda \subseteq \{1, \dots, n\}$ with $|\Lambda| = s$. Set $\mathbb{R}^\Lambda := \{x \in \mathbb{R}^n : x \text{ is supported inside } \Lambda\}$ and $S^\Lambda := \{x \in \mathbb{R}^\Lambda : \|x\| = 1\}$. It is well known that there exists a subset $Q_{\Lambda, \epsilon} \subseteq S^\Lambda$ such that $|Q_{\Lambda, \epsilon}| \leq (24/\epsilon)^s$ and $S^\Lambda \subseteq \cup_{\zeta \in Q_{\Lambda, \epsilon}} \{x \in \mathbb{R}^n : \|x - \zeta\| \leq \epsilon/8\}$. By Theorem 3.5, with probability at least $1 - 3(24/\epsilon)^s e^{-\alpha(\epsilon^2/16 - \epsilon^3/24)m}$, we have

$$(5.1) \quad \sqrt{1 - \epsilon/2} \|v\| \leq \frac{1}{\sqrt{m}} \|A_T v\| \leq \sqrt{1 + \epsilon/2}, \quad \forall T \in T_{\epsilon/2, \alpha} \text{ and } v \in Q_{\Lambda, \epsilon}.$$

We next consider the case where A satisfies (5.1). Define $\lambda := \sup\{\frac{1}{\sqrt{m}} \|A_T x\| : x \in S^\Lambda, T \in T_{\epsilon/2, \alpha}\}$. For every $x \in S^\Lambda$, there exists $v_x \in Q_{\Lambda, \epsilon}$ such that $\|x - v_x\| \leq \epsilon/8$ and hence,

$$\frac{1}{\sqrt{m}} \|A_T x\| \leq \frac{1}{\sqrt{m}} \|A_T v_x\| + \frac{1}{\sqrt{m}} \|A(x - v_x)\| \leq \sqrt{1 + \epsilon/2} + \lambda \|x - v_x\| \leq \sqrt{1 + \epsilon/2} + \lambda \epsilon/8.$$

By the definition of λ , we must have $\lambda \leq \sqrt{1 + \epsilon/2} + \lambda \epsilon/8$, which implies that

$$\lambda \leq \frac{\sqrt{1 + \epsilon/2}}{1 - \epsilon/8} \leq \sqrt{1 + \epsilon}$$

for all $0 < \epsilon < 1$. Therefore, for all $x \in \mathbb{R}^\Lambda$ and $T \in T_{\epsilon/2, \alpha}$, $\frac{1}{\sqrt{m}} \|A_T x\| \leq \lambda \|x\| \leq \sqrt{1 + \epsilon} \|x\|$ and

$$\frac{1}{\sqrt{m}} \|A_T x\| \geq \frac{1}{\sqrt{m}} \|A_T v_x\| - \frac{1}{\sqrt{m}} \|A_T(x - v_x)\| \geq \sqrt{1 - \frac{\epsilon}{2}} - \lambda \frac{\epsilon}{8} \geq \sqrt{1 - \frac{\epsilon}{2}} - \frac{\epsilon}{8} \sqrt{1 + \epsilon} \geq \sqrt{1 - \epsilon},$$

where the last inequality holds for all $0 \leq \epsilon \leq 1$. Thus, with probability at least $1 - 3(24/\epsilon)^s e^{-\alpha(\epsilon^2/16 - \epsilon^3/24)m}$,

$$(5.2) \quad (1 - \epsilon) \|x\|^2 \leq \frac{1}{m} \|A_T x\|^2 \leq (1 + \epsilon) \|x\|^2, \quad \forall x \in \mathbb{R}^\Lambda, T \in T_{\epsilon/2, \alpha}.$$

Note that there are total $\binom{n}{s} \leq (en/s)^s$ such subsets Λ . Therefore, (5.2) holds for every such subset Λ . By union bounds, (1.13) holds with probability at least $1 - 3(\frac{24en}{\epsilon s})^s e^{-\alpha(\epsilon^2/16 - \epsilon^3/24)m} > 0$ by our assumption $s \ln \frac{24en}{\epsilon s} < \alpha(\epsilon^2/16 - \epsilon^3/24)m - \ln 3$. \square

Proof of Corollary 1.5. The condition $0 < \alpha < \frac{\pi}{12}(1 - \beta)^2 h_\beta$ guarantees that $0 < \theta < \infty$, while the condition $m \geq \frac{1}{1 - \beta}$ guarantees $0 < \omega < \infty$ (if $m = \frac{1}{1 - \beta}$, then $0 \ln \frac{\epsilon}{0}$ is understood as $\lim_{x \rightarrow 0^+} x \ln \frac{\epsilon}{x} = 0$.)

By the left-hand inequality in (4.9) of Corollary 4.3, for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 1$, we have $\mathbb{P}(\Omega_{[\theta, \infty], \beta}) \geq 1 - e^{-\alpha m}$. By (4.12) in Corollary 4.4 and $\beta - \frac{1}{m} \leq \gamma \leq \beta$, noting that $(1 - x) \ln \frac{\epsilon}{1 - x}$ is a decreasing function on $(0, 1)$, we deduce that $\mathbb{P}(\Omega_{[0, \omega], \beta}) \geq 1 - e^{-\alpha m}$. Consequently, we have

$$\mathbb{P} \left\{ \theta \|p_j - p_k\|^2 \leq \frac{1}{m} \|A_T p_j - A_T p_k\|^2 \leq \omega \|p_j - p_k\|^2, \forall T \subseteq \{1, \dots, m\}, |T^c| \leq \beta m \right\} \geq 1 - 2e^{-\alpha m}, \quad \forall m \in \mathbb{N}$$

for every $j, k = 1, \dots, N$. Since there are $\binom{N}{2} = \frac{N(N-1)}{2}$ pairs $\{p_j, p_k\}$ with $j \neq k, j, k = 1, \dots, N$, we conclude that (1.25) holds with probability at least $1 - N(N-1)e^{-\alpha m} > 0$ for all $m \in \mathbb{N}$ by $m > \frac{1}{\alpha} \ln \frac{1}{N(N-1)}$. \square

Proof of Corollary 1.6. We use the same notation as in the proof of Corollary 1.3. Define $T_{\leq \beta} := \{T \subseteq \{1, \dots, m\} : |T^c| \leq \beta m\}$. By Corollaries 4.3 and 4.4, with probability at least $1 - 2e^{-\alpha m}$,

$$(5.3) \quad \sqrt{\theta(1 - \epsilon/2)} \|v\| \leq \frac{1}{\sqrt{m}} \|A_T v\| \leq \sqrt{\omega(1 + \epsilon/2)}, \quad \forall T \in T_{\leq \beta}.$$

We next consider the case where A satisfies (5.3). Define $\lambda := \sup\{\frac{1}{\sqrt{m}}\|A_T x\| : x \in S^\Lambda, T \in T_{\leq \beta}\}$. For every $x \in S^\Lambda$, there exists $v_x \in Q_{\Lambda, \epsilon}$ such that $\|x - v_x\| \leq \epsilon/8$ and hence,

$$\frac{1}{\sqrt{m}}\|A_T x\| \leq \frac{1}{\sqrt{m}}\|A_T v_x\| + \frac{1}{\sqrt{m}}\|A(x - v_x)\| \leq \sqrt{\omega(1 + \epsilon/2)} + \lambda\|x - v_x\| \leq \sqrt{\omega(1 + \epsilon/2)} + \lambda\epsilon/8.$$

By the definition of λ , we must have $\lambda \leq \sqrt{\omega(1 + \epsilon/2)} + \lambda\epsilon/8$, from which we have $\lambda \leq \sqrt{\omega(1 + \epsilon/2)}/(1 - \epsilon/8) \leq \sqrt{\omega(1 + \epsilon)}$ for all $0 < \epsilon < 1$. Therefore, for all $x \in \mathbb{R}^\Lambda$ and $T \in T_{\leq \omega}$, $\frac{1}{\sqrt{m}}\|A_T x\| \leq \lambda\|x\| \leq \sqrt{\omega(1 + \epsilon)}\|x\|$ and

$$\begin{aligned} \frac{1}{\sqrt{m}}\|A_T x\| &\geq \frac{1}{\sqrt{m}}\|A_T v_x\| - \frac{1}{\sqrt{m}}\|A_T(x - v_x)\| \geq \sqrt{\theta}\sqrt{1 - \epsilon/2} - \lambda\epsilon/8 \\ &\geq \sqrt{\theta}\left(\sqrt{1 - \epsilon/2} - \sqrt{\omega/\theta}\sqrt{1 + \epsilon}(\epsilon/8)\right) \geq \sqrt{\theta}\left(\sqrt{1 - \epsilon/2} - \sqrt{1 + \epsilon}(\epsilon/8)\right) \geq \sqrt{\theta}\sqrt{1 - \epsilon}, \end{aligned}$$

where we used the fact that $\omega/\theta \geq 1$ by $0 < \theta \leq \omega$. Thus, with probability at least $1 - 2(24/\epsilon)^s e^{-\alpha m}$,

$$(5.4) \quad \theta(1 - \epsilon)\|x\|^2 \leq \frac{1}{m}\|A_T x\|^2 \leq \omega(1 + \epsilon)\|x\|^2, \quad \forall x \in \mathbb{R}^\Lambda, T \in T_{\leq \beta}.$$

Note that there are total $\binom{n}{s} \leq (en/s)^s$ such subsets Λ . Therefore, (5.4) holds for every such subset Λ . Hence, (1.13) holds with probability at least $1 - 2(\frac{24en}{\epsilon s})^s e^{-\alpha m} > 0$ by $s \ln \frac{24en}{\epsilon s} < \alpha m - \ln 2$. \square

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